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Some new Milne-type inequalities

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Abstract

Inequalities play a main role in pure and applied mathematics. In this paper, we prove a generalization of Milne inequality for any measure space. The argument in the proof of this inequality allows us to obtain other Milne-type inequalities. Also, we improve the discrete version of Milne inequality, which holds for any positive value of the parameter p. Finally, we present a Milne-type inequality in the fractional context.

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1 Introduction

The study of classical inequalities in the context of generalized integral operators has gained significant traction in recent years due to their profound implications in differential equations and applied mathematics. Let's briefly overview some of these important inequalities and their applications: Gronwall's inequality provides estimations on the growth of solutions, crucial for proving existence and uniqueness theorems; Chebyshev's inequality is often used in probability theory and statistics to derive bounds on distributions; Hermite–Hadamard-type inequality is utilized in optimization and numerical analysis to approximate integrals; Hardy-type inequality is widely used in harmonic analysis and the study of function spaces; and Opial-type inequality is used in the analysis of differential equations and stability problems (see, e.g., [7, 8, 10, 15, 20–22, 25, 26]).

Rosseland studied the mean stellar absorption coefficient in order to calculate the net flux of radiation. Milne improved this study in [19], by proving the celebrated Milne inequality in [19] (see Proposition 2).

Milne-type inequalities play a pivotal role in various cutting-edge applications by providing reliable error bounds for numerical integration. Their utility spans across multiple disciplines, ensuring the accuracy and robustness of computational methods in modern science, engineering and complex systems [5, 11, 24]. For instance, in big data analytics, numerical integration is essential for estimating probability distributions and density functions. It is important to highlight the theoretical importance of Milne's inequality, which was studied by Hardy, Littlewood and Pólya in their seminal book on inequalities [16]. The theoretical applications of Milne's inequality are extended to approximation theory, geometry and mathematical physics [2–4, 16].

Recent generalizations of Milne-type inequalities have significantly expanded their applicability, particularly through the introduction of fractional calculus. These generalizations incorporate various fractional integral operators, providing refined error bounds and new theoretical insights [9, 12, 13, 17]. One notable development involves the use of tempered fractional integrals. Researchers have extended Milne-type inequalities to fractional settings, incorporating Riemann–Liouville integrals and tempered fractional integrals. These advances enable more precise error estimations in numerical integration, particularly in applications requiring high accuracy, such as big data analytics and financial modeling [24]. Additionally, new inequalities have been established for functions defined by different convexity conditions, such as (s,m)-convex functions. These generalizations offer enhanced flexibility and applicability in various mathematical and engineering contexts [1, 23].

These advancements demonstrate the ongoing evolution of Milne-type inequalities, reinforcing their importance in modern computational methods and broadening their scope of application across diverse scientific disciplines.

In this work, we obtain new Milne-type inequalities, which include a generalization of Milne inequality (see Theorem 3) that improves this classical result in several ways:

- We replace Riemann integral with Lebesgue integral.
- We replace the interval $(0,\infty)$ with any measurable space X.
- We replace the absolutely continuous probability measure $\phi(x)\,dx$ with any measure μ (this measure does not even need to be σ-finite).
- We replace the Riemann-integrable functions with measurable functions (they do not need to be Lebesgue integrable).
- The functions can take the values 0 and ∞ on any measurable set.
- We allow finite and infinite sums.

The argument in the proof of Theorem 3 is so flexible that allows to prove other Milne-type inequalities as Theorem 8. Also, we improve the discrete version of Milne inequality, which holds for every positive value of the parameter p, see Theorems 5 and 6. Furthermore, Theorem 3 allows us to extend Milne's inequality to the fractional context (global and local). Thus, we present a Milne-type inequality in the context of the generalized Riemann–Liouville-type integral operators defined in [6], which include most of the known Riemann–Liouville-type integral operators.

2 Milne-type inequalities

Milne proved in 1925 the two following discrete and continuous versions of a useful inequality [19]:

Proposition 1 *The following inequality holds for every $a_i, b_i > 0$ for $1 \le i \le n$:*

$$\sum_{i=1}^{n} a_i \cdot \sum_{i=1}^{n} b_i \ge \sum_{i=1}^{n} (a_i + b_i) \cdot \sum_{i=1}^{n} \frac{a_i b_i}{a_i + b_i}. \tag{1}$$

Remark 1 Since

$$\frac{a_i b_i}{a_i + b_i} \le \frac{1}{4}(a_i + b_i),$$

the conclusion of Proposition 1 also holds for every $a_i, b_i \geq 0$ with the convention $0 \cdot 0/(0+0) = 0$.

Proposition 2 *Let $\phi : (0,\infty) \to [0,\infty)$ be a Riemann integrable function with $\int_0^\infty \phi(x)\,dx = 1$. Let $a_i > 0$ and $f_i : (0,\infty) \to (0,\infty)$ such that ϕ/f_i is a Riemann integrable function on $(0,\infty)$ for $1 \leq i \leq n$. Then,*

$$\frac{1}{\displaystyle\int_0^\infty \frac{\phi(x)\,dx}{a_1 f_1(x) + \cdots + a_n f_n(x)}} \geq \frac{a_1}{\displaystyle\int_0^\infty \frac{\phi(x)\,dx}{f_1(x)}} + \cdots + \frac{a_n}{\displaystyle\int_0^\infty \frac{\phi(x)\,dx}{f_n(x)}}. \tag{2}$$

From now on, we use the usual conventions $1/\infty = 0$ and $1/0 = \infty$.

The following theorem is the main result in this paper, generalizing Proposition 2, the continuous version of Milne inequality.

Theorem 3 *Let μ be a measure on the space X, $a_n \geq 0$ and $f_n : X \to [0,\infty]$ measurable functions for $n \geq 1$. Then,*

$$\frac{1}{\displaystyle\int_X \frac{d\mu(x)}{\sum_{n=1}^\infty a_n f_n(x)}} \geq \sum_{n=1}^\infty \frac{a_n}{\displaystyle\int_X \frac{d\mu(x)}{f_n(x)}}. \tag{3}$$

Proof Let us prove first

$$\frac{1}{\displaystyle\int_X \frac{d\mu(x)}{a_1 f_1(x) + \cdots + a_n f_n(x)}} \geq \frac{a_1}{\displaystyle\int_X \frac{d\mu(x)}{f_1(x)}} + \cdots + \frac{a_n}{\displaystyle\int_X \frac{d\mu(x)}{f_n(x)}} \tag{4}$$

for every $n \geq 1$. If we define $g_i(x) := a_i f_i(x)$, then it suffices to prove that

$$\frac{1}{\displaystyle\int_X \frac{d\mu(x)}{g_1(x) + \cdots + g_n(x)}} \geq \frac{1}{\displaystyle\int_X \frac{d\mu(x)}{g_1(x)}} + \cdots + \frac{1}{\displaystyle\int_X \frac{d\mu(x)}{g_n(x)}}. \tag{5}$$

Assume that (5) holds if $1/g_i \in L^1(X,\mu)$ for every $1 \leq i \leq n$. Now, if $1/g_i \in L^1(X,\mu)$ for $1 \leq i \leq k < n$ and $1/g_i \notin L^1(X,\mu)$ for $k < i \leq n$, then

$$\frac{1}{\displaystyle\int_X \frac{d\mu(x)}{g_i(x)}} = \frac{1}{\infty} = 0$$

for $k < i \leq n$, and so

$$\begin{aligned}
\frac{1}{\displaystyle\int_X \frac{d\mu(x)}{g_1(x) + \cdots + g_n(x)}} &\geq \frac{1}{\displaystyle\int_X \frac{d\mu(x)}{g_1(x) + \cdots + g_k(x)}} \\
&\geq \frac{1}{\displaystyle\int_X \frac{d\mu(x)}{g_1(x)}} + \cdots + \frac{1}{\displaystyle\int_X \frac{d\mu(x)}{g_k(x)}} \\
&= \frac{1}{\displaystyle\int_X \frac{d\mu(x)}{g_1(x)}} + \cdots + \frac{1}{\displaystyle\int_X \frac{d\mu(x)}{g_n(x)}}.
\end{aligned}$$

Hence, without loss of generality, we can assume that $1/g_i \in L^1(X,\mu)$ for every $1 \le i \le n$.

Let us prove (5) in this case by induction on n.

If $n = 1$, then the inequality (5) holds since, in fact, it is an equality.

If $n = 2$, $A(x) := 1/g_1(x)$ and $B(x) := 1/g_2(x)$, then the following inequalities are equivalent

$$\frac{1}{\int_X \frac{d\mu(x)}{g_1(x)+g_2(x)}} \ge \frac{1}{\int_X \frac{d\mu(x)}{g_1(x)}} + \frac{1}{\int_X \frac{d\mu(x)}{g_2(x)}}$$

$$\frac{1}{\int_X \frac{d\mu(x)}{1/A(x)+1/B(x)}} \ge \frac{1}{\int_X A(x)\,d\mu(x)} + \frac{1}{\int_X B(x)\,d\mu(x)}$$

$$\frac{1}{\int_X \frac{A(x)B(x)}{A(x)+B(x)}\,d\mu(x)} \ge \frac{\int_X (A(x)+B(x))\,d\mu(x)}{\int_X A(x)\,d\mu(x)\int_X B(x)\,d\mu(x)}$$

and then, we obtain

$$\int_X A(x)\,d\mu(x)\int_X B(x)\,d\mu(x) \ge \int_X \big(A(x)+B(x)\big)\,d\mu(x)\int_X \frac{A(x)B(x)}{A(x)+B(x)}\,d\mu(x). \tag{6}$$

Note that since $A, B \in L^1(X,\mu)$ and

$$\frac{A(x)B(x)}{A(x)+B(x)} \le \frac{1}{4}\big(A(x)+B(x)\big),$$

the four integrals in (6) are finite, with the convention $0 \cdot 0/(0+0) = 0$.

We are going to construct simple functions $\{s_m\}$ and $\{t_m\}$ approximating A and B, respectively, in such a way that $\{s_m + t_m\}$ approximates $A + B$, $\{s_m t_m\}$ approximates AB, and $\{s_m t_m/(s_m + t_m)\}$ approximates $AB/(A + B)$. A main ingredient in this approach is to also be able to write the functions $s_m + t_m$, $s_m t_m$ and $s_m t_m/(s_m + t_m)$ as simple functions.

Denote by χ_E the characteristic function of the set E, i.e., the function with $\chi_E = 1$ on E and $\chi_E = 0$ on $X \setminus E$. For each $m > 1$ and $0 \le i, j \le m2^m$, let us consider the measurable sets

$$\begin{aligned} A_i &:= \{x \in X : A(x) \in [i2^{-m}, (i+1)2^{-m})\}, \text{ if } 0 \le i < m2^m, \\ A_{m2^m} &:= \{x \in X : A(x) \ge m\}, \\ B_j &:= \{x \in X : B(x) \in [j2^{-m}, (j+1)2^{-m})\}, \text{ if } 0 \le j < m2^m, \\ B_{m2^m} &:= \{x \in X : B(x) \ge m\}, \end{aligned}$$

and the simple functions

$$s_m(x) := \sum_{i=0}^{m2^m} i2^{-m}\chi_{A_i}(x),$$

$$t_m(x) := \sum_{j=0}^{m2^m} j2^{-m}\chi_{B_j}(x).$$

It is clear that $\{s_m\}$ and $\{t_m\}$ are sequences of non-decreasing simple functions satisfying

$$\begin{aligned} 0 \le s_m(x) \le A(x), &\qquad \lim_{m\to\infty} s_m(x) = A(x), \\ 0 \le t_m(x) \le B(x), &\qquad \lim_{m\to\infty} t_m(x) = B(x), \end{aligned}$$

for every $x \in X$. Thus, monotone convergence theorem provides

$$\begin{aligned} \lim_{m\to\infty} \int_X s_m(x)\,d\mu(x) &= \int_X A(x)\,d\mu(x), \\ \lim_{m\to\infty} \int_X t_m(x)\,d\mu(x) &= \int_X B(x)\,d\mu(x), \\ \lim_{m\to\infty} \int_X \big(s_m(x) + t_m(x)\big)\,d\mu(x) &= \int_X \big(A(x) + B(x)\big)\,d\mu(x). \end{aligned}$$

Since $\{A_i\}_{i=0}^{m2^m}$, $\{B_j\}_{j=0}^{m2^m}$ and $\{A_i \cap B_j\}_{i,j=0}^{m2^m}$ are partitions of X, we have

$$\begin{aligned} s_m(x) &= \sum_{i=0}^{m2^m} i2^{-m}\chi_{A_i}(x) = \sum_{i,j=0}^{m2^m} i2^{-m}\chi_{A_i}(x)\chi_{B_j}(x), \\ t_m(x) &= \sum_{j=0}^{m2^m} j2^{-m}\chi_{B_j}(x) = \sum_{i,j=0}^{m2^m} j2^{-m}\chi_{A_i}(x)\chi_{B_j}(x), \\ \int_X s_m(x)\,d\mu(x) &= \sum_{i=0}^{m2^m} i2^{-m}\,\mu(A_i) = \sum_{i,j=0}^{m2^m} i2^{-m}\,\mu(A_i \cap B_j), \\ \int_X t_m(x)\,d\mu(x) &= \sum_{j=0}^{m2^m} j2^{-m}\,\mu(B_j) = \sum_{i,j=0}^{m2^m} j2^{-m}\,\mu(A_i \cap B_j), \\ \int_X \big(s_m(x) + t_m(x)\big)\,d\mu(x) &= \sum_{i,j=0}^{m2^m} \big(i2^{-m} + j2^{-m}\big)\mu(A_i \cap B_j). \end{aligned}$$

We also have

$$\begin{aligned} \frac{s_m(x)\,t_m(x)}{s_m(x) + t_m(x)} &= \frac{\displaystyle\sum_{i,j=0}^{m2^m} i2^{-m}j2^{-m}\chi_{A_i}(x)\chi_{B_j}(x)}{\displaystyle\sum_{i,j=0}^{m2^m} \big(i2^{-m} + j2^{-m}\big)\chi_{A_i\cap B_j}(x)} \\ &= \sum_{i,j=0}^{m2^m} \frac{i2^{-m}j2^{-m}}{i2^{-m} + j2^{-m}}\,\chi_{A_i\cap B_j}(x) \end{aligned}$$

and integrating on both sides of the equality, we obtain

$$\int_X \frac{s_m(x)\,t_m(x)}{s_m(x)+t_m(x)}\,d\mu(x) = \sum_{i,j=0}^{m2^m} \frac{i2^{-m}j2^{-m}}{i2^{-m}+j2^{-m}}\,\mu(A_i\cap B_j). \tag{7}$$

Hence, the following inequalities are equivalent:

$$\int_X s_m(x)\,d\mu(x)\int_X t_m(x)\,d\mu(x) \ge \int_X \big(s_m(x)+t_m(x)\big)\,d\mu(x)\int_X \frac{s_m(x)\,t_m(x)}{s_m(x)+t_m(x)}\,d\mu(x),$$

$$\sum_{i,j=0}^{m2^m} i2^{-m}\,\mu(A_i\cap B_j)\sum_{i,j=0}^{m2^m} j2^{-m}\,\mu(A_i\cap B_j) \ge \sum_{i,j=0}^{m2^m}\big(i2^{-m}+j2^{-m}\big)\mu(A_i\cap B_j)\sum_{i,j=0}^{m2^m}\frac{i2^{-m}j2^{-m}}{i2^{-m}+j2^{-m}}\,\mu(A_i\cap B_j),$$

$$\sum_{i,j=0}^{m2^m} i\,\mu(A_i\cap B_j)\sum_{i,j=0}^{m2^m} j\,\mu(A_i\cap B_j) \ge \sum_{i,j=0}^{m2^m}\big(i\,\mu(A_i\cap B_j)+j\,\mu(A_i\cap B_j)\big)\sum_{i,j=0}^{m2^m}\frac{i\,\mu(A_i\cap B_j)\,j\,\mu(A_i\cap B_j)}{i\,\mu(A_i\cap B_j)+j\,\mu(A_i\cap B_j)},$$

and this last inequality holds by Proposition 1 and Remark 1. Hence, in order to prove the case $n=2$ it suffices to prove that

$$\lim_{m\to\infty}\int_X \frac{s_m(x)\,t_m(x)}{s_m(x)+t_m(x)}\,d\mu(x) = \int_X \frac{A(x)B(x)}{A(x)+B(x)}\,d\mu(x).$$

Since $\{s_m\}$ and $\{t_m\}$ increases to A and B respectively, then

$$\frac{s_m(x)\,t_m(x)}{s_m(x)+t_m(x)} = \frac{1}{1/s_m(x)+1/t_m(x)}$$

increases to

$$\frac{A(x)B(x)}{A(x)+B(x)} = \frac{1}{1/A(x)+1/B(x)} \in L^1(X,\mu),$$

and monotone or dominated convergence theorem gives

$$\lim_{m\to\infty}\int_X \frac{s_m(x)\,t_m(x)}{s_m(x)+t_m(x)}\,d\mu(x) = \int_X \frac{A(x)B(x)}{A(x)+B(x)}\,d\mu(x).$$

Finally, assume that (5) holds for $n-1\ge 2$. Then, the induction hypothesis and the previous inequality give

$$\frac{1}{\displaystyle\int_X \frac{d\mu(x)}{g_1(x)+\cdots+g_{n-2}(x)+g_{n-1}(x)+g_n(x)}} \ge \frac{1}{\displaystyle\int_X \frac{d\mu(x)}{g_1(x)}}+\cdots+\frac{1}{\displaystyle\int_X \frac{d\mu(x)}{g_{n-2}(x)}}+\frac{1}{\displaystyle\int_X \frac{d\mu(x)}{g_{n-1}(x)+g_n(x)}}$$

$$\geq \frac{1}{\int_X \frac{d\mu(x)}{g_1(x)}} + \cdots + \frac{1}{\int_X \frac{d\mu(x)}{g_{n-2}(x)}} + \frac{1}{\int_X \frac{d\mu(x)}{g_{n-1}(x)}} + \frac{1}{\int_X \frac{d\mu(x)}{g_n(x)}},$$

which finishes the proof of (5) and so, (4) holds. Since

$$\sum_{n=1}^{\infty} a_n f_n(x) \geq \sum_{n=1}^{N} a_n f_n(x)$$

for every N, we have

$$\frac{1}{\int_X \frac{d\mu(x)}{\sum_{n=1}^{\infty} a_n f_n(x)}} \geq \frac{1}{\int_X \frac{d\mu(x)}{\sum_{n=1}^{N} a_n f_n(x)}} \geq \sum_{n=1}^{N} \frac{a_n}{\int_X \frac{d\mu(x)}{f_n(x)}}$$

for every N. The desired inequality follows by taking limits as N goes to ∞. □

Corollary 4 *Let μ be a measure on the space X, $a_i \geq 0$ and $f_i : X \to [0,\infty]$ measurable functions for $1 \leq i \leq n$. Then,*

$$\frac{1}{\int_X \frac{d\mu(x)}{a_1 f_1(x) + \cdots + a_n f_n(x)}} \geq \frac{a_1}{\int_X \frac{d\mu(x)}{f_1(x)}} + \cdots + \frac{a_n}{\int_X \frac{d\mu(x)}{f_n(x)}}. \tag{8}$$

We show now a generalization of Proposition 1, the discrete version of Milne inequality, which holds for any positive value of the parameter p.

Theorem 5 *Let $p > 0$ and $a_i, b_i \geq 0$ for $1 \leq i \leq n$. Then,*

$$\left(\sum_{i=1}^{n} a_i^p\right)^{1/p} \left(\sum_{i=1}^{n} b_i^p\right)^{1/p} \geq \left(\left(\sum_{i=1}^{n} a_i^p\right)^{1/p} + \left(\sum_{i=1}^{n} b_i^p\right)^{1/p}\right) \left(\sum_{i=1}^{n} \left(\frac{a_i b_i}{a_i + b_i}\right)^p\right)^{1/p} \tag{9}$$

with the convention $0 \cdot 0/(0+0) = 0$. The equality in the bound (9) is attained if and only if there exist constants $\lambda, \mu \geq 0$ with $\lambda + \mu > 0$ and $\lambda a_i = \mu b_i$ for every $1 \leq i \leq n$.

Proof If $\sum_{i=1}^{n} a_i^p = 0$, then the desired inequality is $0 \geq 0$, and we have $1 \cdot a_i = 0 \cdot b_i$ for every $1 \leq i \leq n$. By symmetry, a similar statement holds if $\sum_{i=1}^{n} b_i^p = 0$.

Let us consider the case $\sum_{i=1}^{n} a_i^p > 0$, $\sum_{i=1}^{n} b_i^p > 0$. It suffices to prove that

$$f(a_1,\dots,a_n,b_1,\dots,b_n) := \sum_{i=1}^{n} \left(\frac{a_i b_i}{a_i + b_i}\right)^p \leq \frac{AB}{(A^{1/p} + B^{1/p})^p}, \tag{10}$$

when

$$g_1(a_1,\dots,a_n,b_1,\dots,b_n) := \sum_{i=1}^{n} a_i^p = A > 0,$$

$$g_2(a_1,\dots,a_n,b_1,\dots,b_n) := \sum_{i=1}^{n} b_i^p = B > 0,$$

$$a_i \geq 0, \quad b_i \geq 0, \quad 1 \leq i \leq n,$$

and the equality in the bound is attained if and only if there exist constants $\lambda, \mu \geq 0$ with $\lambda + \mu > 0$ and $\lambda a_i = \mu b_i$ for every $1 \leq i \leq n$.

Note that f is a continuous function even if $a_i = 0$ and/or $b_i = 0$ for some $1 \leq i \leq n$, since

$$\frac{a_i b_i}{a_i + b_i} \leq \frac{1}{4}(a_i + b_i).$$

Let us prove (10) by induction on n.

If $n = 1$, then

$$f(a_1, b_1) = \Big(\frac{a_1 b_1}{a_1 + b_1}\Big)^p = \Big(\frac{A^{1/p} B^{1/p}}{A^{1/p} + B^{1/p}}\Big)^p = \frac{AB}{(A^{1/p} + B^{1/p})^p},$$

and $a_1 B^{1/p} = a_1 b_1 = b_1 A^{1/p}$.

Assume that (10) holds for $n - 1 \geq 1$ and let us show it for n.

If there exists a maximum point with $a_i = b_i = 0$ for some $1 \leq i \leq n$, by symmetry we can assume that $i = n$, and then the induction hypothesis gives the result. Hence, we can assume that $(a_i, b_i) \neq (0,0)$ for every $1 \leq i \leq n$.

Let us prove now that any maximum point satisfies $a_i, b_i > 0$ for every $1 \leq i \leq n$.

If there exists a maximum point with $a_i b_i = 0$ for every $1 \leq i \leq n$, then $f(a_1, \dots, a_n, b_1, \dots, b_n) = 0$, a contradiction. Hence, we can assume that $a_i, b_i > 0$ for some $1 \leq i \leq n$. By symmetry, we can assume that $a_1, b_1 > 0$.

If there exists a maximum point $(a_1, \dots, a_n, b_1, \dots, b_n)$ with $a_i b_i = 0$ for some $1 < i \leq n$, by symmetry, we can assume that $i = n$, $a_n = 0$ and $b_n > 0$. We obtain the point $(a_1, \dots, a_n, b_1', b_2 \dots, b_{n-1}, 0)$ from $(a_1, \dots, a_n, b_1, \dots, b_n)$ with $b_1' = (b_1^p + b_n^p)^{1/p} > b_1$. Note that

$$(b_1')^p + b_2^p + \cdots + b_{n-1}^p + 0^p = b_1^p + b_n^p + b_2^p + \cdots + b_{n-1}^p = B$$

and, since $b_1' > b_1$ and $a_1 > 0$,

$$\Big(\frac{a_1 b_1'}{a_1 + b_1'}\Big)^p = \Big(\frac{1}{1/a_1 + 1/b_1'}\Big)^p > \Big(\frac{1}{1/a_1 + 1/b_1}\Big)^p = \Big(\frac{a_1 b_1}{a_1 + b_1}\Big)^p.$$

Therefore, since $a_n = 0$,

$$\begin{aligned} f(a_1, \dots, a_n, b_1', b_2 \dots, b_{n-1}, 0) &= \Big(\frac{a_1 b_1'}{a_1 + b_1'}\Big)^p + \sum_{i=2}^{n-1} \Big(\frac{a_i b_i}{a_i + b_i}\Big)^p \\ &> \Big(\frac{a_1 b_1}{a_1 + b_1}\Big)^p + \sum_{i=2}^{n-1} \Big(\frac{a_i b_i}{a_i + b_i}\Big)^p = f(a_1, \dots, a_n, b_1, \dots, b_n), \end{aligned}$$

a contradiction.

Hence, any maximum point satisfies $a_i, b_i > 0$ for every $1 \leq i \leq n$.

The method of Lagrange multipliers gives that in each critical point there exist $\lambda_1, \lambda_2 \in \mathbb{R}$ such that

$$\nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2,$$

that is

$$p\left(\frac{a_ib_i}{a_i+b_i}\right)^{p-1}\frac{b_i^2}{(a_i+b_i)^2}=\lambda_1 p\,a_i^{p-1},$$
$$p\left(\frac{a_ib_i}{a_i+b_i}\right)^{p-1}\frac{a_i^2}{(a_i+b_i)^2}=\lambda_2 p\,b_i^{p-1},$$

for every $1\le i\le n$. Since $a_i,b_i>0$ for every $1\le i\le n$, we have $\lambda_1,\lambda_2>0$. Thus, the above equations are equivalent to

$$\frac{b_i}{a_i+b_i}=\lambda_1^{1/(p+1)},$$
$$\frac{a_i}{a_i+b_i}=\lambda_2^{1/(p+1)},$$

for every $1\le i\le n$. Thus,

$$b_i=\left(\frac{\lambda_1}{\lambda_2}\right)^{1/(p+1)}a_i=:c\,a_i,$$

for each $1\le i\le n$, and so, $c>0$ and

$$B=\sum_{i=1}^n b_i^p=c^p\sum_{i=1}^n a_i^p=c^pA,\qquad c=B^{1/p}/A^{1/p}.$$

We have at these critical points

$$\begin{aligned}f(a_1,\dots,a_n,b_1,\dots,b_n)&=\sum_{i=1}^n\left(\frac{a_ib_i}{a_i+b_i}\right)^p=\sum_{i=1}^n\left(\frac{a_ic\,a_i}{a_i+c\,a_i}\right)^p=\left(\frac{c}{1+c}\right)^p\sum_{i=1}^n a_i^p\\&=\left(\frac{B^{1/p}/A^{1/p}}{1+B^{1/p}/A^{1/p}}\right)^pA=\frac{AB}{(A^{1/p}+B^{1/p})^p},\end{aligned}$$

and this finishes the proof. □

We have the following consequence of Theorem 5.

Theorem 6 *Let $p\ge 1$ and $a_i,b_i\ge 0$ for $1\le i\le n$. Then,*

$$\left(\sum_{i=1}^n a_i^p\right)\left(\sum_{i=1}^n b_i^p\right)\ge\sum_{i=1}^n(a_i+b_i)^p\sum_{i=1}^n\left(\frac{a_ib_i}{a_i+b_i}\right)^p,\tag{11}$$

with the convention $0\cdot 0/(0+0)=0$. The equality in the bound (11) is attained if and only if there exist constants $\lambda,\mu\ge 0$ with $\lambda+\mu>0$ and $\lambda a_i=\mu b_i$ for every $1\le i\le n$.

Proof Since triangle inequality gives

$$\left(\sum_{i=1}^n(a_i+b_i)^p\right)^{1/p}\le\left(\sum_{i=1}^n a_i^p\right)^{1/p}+\left(\sum_{i=1}^n b_i^p\right)^{1/p}$$

for any $p\ge 1$, Theorem 5 allows to obtain the desired inequality.

If the equality in the bound is attained, then the inequality in Theorem 5 is attained also and so, there exist constants $\lambda, \mu \geq 0$ with $\lambda + \mu > 0$ and $\lambda a_i = \mu b_i$ for every $1 \leq i \leq n$.

Assume now that there exist constants $\lambda, \mu \geq 0$ with $\lambda + \mu > 0$ and $\lambda a_i = \mu b_i$ for every $1 \leq i \leq n$. If $\mu = 0$, then $a_i = 0$ for every $1 \leq i \leq n$ and the inequality is $0 \geq 0$. If $\mu > 0$, then $b_i = ca_i$ for every $1 \leq i \leq n$, with $c = \lambda/\mu \geq 0$. Thus,

$$\left(\sum_{i=1}^{n} a_i^p\right)\left(\sum_{i=1}^{n} b_i^p\right) = \left(\sum_{i=1}^{n} a_i^p\right)\left(\sum_{i=1}^{n} c^p a_i^p\right) = c^p\left(\sum_{i=1}^{n} a_i^p\right)^2$$

and

$$\begin{aligned}\sum_{i=1}^{n}(a_i + b_i)^p \sum_{i=1}^{n}\left(\frac{a_i b_i}{a_i + b_i}\right)^p &= \sum_{i=1}^{n}(a_i + ca_i)^p \sum_{i=1}^{n}\left(\frac{a_i ca_i}{a_i + ca_i}\right)^p \\ &= (1+c)^p \sum_{i=1}^{n} a_i^p \left(\frac{c}{1+c}\right)^p \sum_{i=1}^{n} a_i^p \\ &= c^p\left(\sum_{i=1}^{n} a_i^p\right)^2 = \left(\sum_{i=1}^{n} a_i^p\right)\left(\sum_{i=1}^{n} b_i^p\right).\end{aligned}$$

This finishes the proof. □

Corollary 7 *Let $p \geq 1$ and $a_i, b_i, w_i \geq 0$ for $1 \leq i \leq n$. Then,*

$$\left(\sum_{i=1}^{n} a_i^p w_i\right)\left(\sum_{i=1}^{n} b_i^p w_i\right) \geq \sum_{i=1}^{n}(a_i + b_i)^p w_i \sum_{i=1}^{n}\left(\frac{a_i b_i}{a_i + b_i}\right)^p w_i,$$

with the convention $0 \cdot 0/(0+0) = 0$.

Proof It suffices to replace a_i and b_i with $a_i w_i^{1/p}$ and $b_i w_i^{1/p}$, respectively, in Theorem 6. □

The approximation by simple functions in the proof of Theorem 3 is so flexible that allows to prove other Milne-type inequalities as the following.

Theorem 8 *Let μ be a measure on the space X, $p \geq 1$ and let $A, B : X \to [0, \infty]$ be measurable functions. Then,*

$$\int_X A(x)^p \, d\mu(x) \int_X B(x)^p \, d\mu(x) \geq \int_X \big(A(x) + B(x)\big)^p \, d\mu(x) \int_X \left(\frac{A(x)B(x)}{A(x) + B(x)}\right)^p d\mu(x),$$

with the convention $0 \cdot 0/(0+0) = 0$.

Proof As in the proof of Theorem 3, consider the simple functions

$$s_m(x) := \sum_{i=0}^{m2^m} i2^{-m} \chi_{A_i}(x),$$

$$t_m(x) := \sum_{j=0}^{m2^m} j2^{-m} \chi_{B_j}(x).$$

Since $\{s_m\}$ and $\{t_m\}$ are sequences of non-decreasing simple functions converging to A and B, respectively, monotone convergence theorem provides

$$\lim_{m\to\infty}\int_X s_m(x)^p\,d\mu(x)=\int_X A(x)^p\,d\mu(x),$$
$$\lim_{m\to\infty}\int_X t_m(x)^p\,d\mu(x)=\int_X B(x)^p\,d\mu(x),$$
$$\lim_{m\to\infty}\int_X \bigl(s_m(x)+t_m(x)\bigr)^p\,d\mu(x)=\int_X \bigl(A(x)+B(x)\bigr)^p\,d\mu(x).$$

Since $\{A_i\}_{i=0}^{m2^m}$, $\{B_j\}_{j=0}^{m2^m}$ and $\{A_i\cap B_j\}_{i,j=0}^{m2^m}$ are partitions of X, we have

$$s_m(x)^p=\sum_{i=0}^{m2^m}(i2^{-m})^p\chi_{A_i}(x)=\sum_{i,j=0}^{m2^m}(i2^{-m})^p\chi_{A_i}(x)\chi_{B_j}(x),$$
$$t_m(x)^p=\sum_{j=0}^{m2^m}(j2^{-m})^p\chi_{B_j}(x)=\sum_{i,j=0}^{m2^m}(j2^{-m})^p\chi_{A_i}(x)\chi_{B_j}(x),$$
$$\int_X s_m(x)^p\,d\mu(x)=\sum_{i,j=0}^{m2^m}(i2^{-m})^p\,\mu(A_i\cap B_j),$$
$$\int_X t_m(x)^p\,d\mu(x)=\sum_{i,j=0}^{m2^m}(j2^{-m})^p\,\mu(A_i\cap B_j),$$
$$\int_X \bigl(s_m(x)+t_m(x)\bigr)^p\,d\mu(x)=\sum_{i,j=0}^{m2^m}\bigl(i2^{-m}+j2^{-m}\bigr)^p\mu(A_i\cap B_j).$$

We also have

$$\Bigl(\frac{s_m(x)\,t_m(x)}{s_m(x)+t_m(x)}\Bigr)^p=\frac{\sum_{i,j=0}^{m2^m}\bigl(i2^{-m}j2^{-m}\bigr)^p\chi_{A_i}(x)\chi_{B_j}(x)}{\sum_{i,j=0}^{m2^m}\bigl(i2^{-m}+j2^{-m}\bigr)^p\chi_{A_i\cap B_j}(x)}$$
$$=\sum_{i,j=0}^{m2^m}\Bigl(\frac{i2^{-m}j2^{-m}}{i2^{-m}+j2^{-m}}\Bigr)^p\,\chi_{A_i\cap B_j}(x)$$

and integrating on both sides of the equality, we obtain

$$\int_X\Bigl(\frac{s_m(x)\,t_m(x)}{s_m(x)+t_m(x)}\Bigr)^p\,d\mu(x)=\sum_{i,j=0}^{m2^m}\Bigl(\frac{i2^{-m}j2^{-m}}{i2^{-m}+j2^{-m}}\Bigr)^p\,\mu(A_i\cap B_j).$$

Hence, the following inequalities are equivalent:

$$\int_X s_m(x)^p\,d\mu(x)\int_X t_m(x)^p\,d\mu(x)$$
$$\geq\int_X\bigl(s_m(x)+t_m(x)\bigr)^p\,d\mu(x)\int_X\Bigl(\frac{s_m(x)\,t_m(x)}{s_m(x)+t_m(x)}\Bigr)^p\,d\mu(x),$$

$$\sum_{i,j=0}^{m2^m} \left(i2^{-m}\right)^p \mu(A_i \cap B_j) \sum_{i,j=0}^{m2^m} \left(j2^{-m}\right)^p \mu(A_i \cap B_j)$$
$$\geq \sum_{i,j=0}^{m2^m} \left(i2^{-m} + j2^{-m}\right)^p \mu(A_i \cap B_j) \sum_{i,j=0}^{m2^m} \left(\frac{i2^{-m} j2^{-m}}{i2^{-m} + j2^{-m}}\right)^p \mu(A_i \cap B_j),$$

and this last inequality holds by Corollary 7. Hence, it suffices to prove that

$$\lim_{m\to\infty} \int_X \left(\frac{s_m(x)\, t_m(x)}{s_m(x) + t_m(x)}\right)^p d\mu(x) = \int_X \left(\frac{A(x)B(x)}{A(x) + B(x)}\right)^p d\mu(x).$$

Since $\{s_m\}$ and $\{t_m\}$ increases to A and B respectively, then

$$\left(\frac{s_m(x)\, t_m(x)}{s_m(x) + t_m(x)}\right)^p = \left(\frac{1}{1/s_m(x) + 1/t_m(x)}\right)^p$$

increases to

$$\left(\frac{A(x)B(x)}{A(x) + B(x)}\right)^p = \left(\frac{1}{1/A(x) + 1/B(x)}\right)^p,$$

and monotone convergence theorem gives

$$\lim_{m\to\infty} \int_X \left(\frac{s_m(x)\, t_m(x)}{s_m(x) + t_m(x)}\right)^p d\mu(x) = \int_X \left(\frac{A(x)B(x)}{A(x) + B(x)}\right)^p d\mu(x). \qquad \square$$

3 Applications to general fractional integral of Riemann–Liouville type

The authors give the definition of a general fractional integral in [6].

Definition 1 Let $a < b$ and $\alpha \in \mathbb{R}^+$. Let $g : [a,b] \to \mathbb{R}$ be a positive function on $(a,b]$ with continuous positive derivative on (a,b), and $G : [0, g(b) - g(a)] \times (0,\infty) \to \mathbb{R}$ a continuous function, which is positive on $(0, g(b) - g(a)] \times (0,\infty)$. Let us define the function $T : [a,b] \times [a,b] \times (0,\infty) \to \mathbb{R}$ by

$$T(t,s,\alpha) = \frac{G\big(|g(t) - g(s)|, \alpha\big)}{g'(s)}. \tag{12}$$

The *right and left integral operators*, denoted respectively by J^{α}_{T,a^+} and J^{α}_{T,b^-}, are defined for each measurable function f on $[a,b]$ as

$$J^{\alpha}_{T,a^+} f(t) = \int_a^t \frac{f(s)}{T(t,s,\alpha)}\, ds, \tag{13}$$
$$J^{\alpha}_{T,b^-} f(t) = \int_t^b \frac{f(s)}{T(t,s,\alpha)}\, ds, \tag{14}$$

with $t \in [a,b]$.

We say that $f \in L^1_T[a,b]$ if $J^{\alpha}_{T,a^+}|f|(t) < \infty$ and $J^{\alpha}_{T,b^-}|f|(t) < \infty$ for every $t \in [a,b]$.

Note that these operators generalize many integral operators associated with some kinds of Riemann–Liouville fractional derivatives. For instance:

1. If we choose

$$g(t) = t, \quad G(x,\alpha) = \Gamma(\alpha)\, x^{1-\alpha}, \quad T(t,s,\alpha) = \Gamma(\alpha)\, |t-s|^{1-\alpha},$$

   then J^{α}_{T,a^+} and J^{α}_{T,b^-} are the right and left Riemann–Liouville fractional integrals (see [14]).
2. If we choose

$$g(t) = \log t, \quad G(x,\alpha) = \Gamma(\alpha)\, x^{1-\alpha}, \quad T(t,s,\alpha) = \Gamma(\alpha)\, t \left| \log \frac{t}{s} \right|^{1-\alpha},$$

   then J^{α}_{T,a^+} and J^{α}_{T,b^-} are the right and left Hadamard fractional integrals.
3. If we choose a function g with the properties in Definition 1 and

$$G(x,\alpha) = \Gamma(\alpha)\, x^{1-\alpha}, \quad T(t,s,\alpha) = \Gamma(\alpha)\, \frac{|g(t)-g(s)|^{1-\alpha}}{g'(s)},$$

   then J^{α}_{T,a^+} and J^{α}_{T,b^-} are the right and left Kilbas–Marichev–Samko fractional integrals (see [18]).

Theorem 3 has the following direct consequence for general fractional integrals of Riemann–Liouville type.

Proposition 9 *Let $a_n \geq 0$ and $f_n : [a,b] \to [0,\infty]$ measurable functions for $n \geq 1$. Then,*

$$\frac{1}{\displaystyle\int_a^b \frac{dx}{T(b,x,\alpha)\sum_{n=1}^{\infty} a_n f_n(x)}} \geq \sum_{n=1}^{\infty} \frac{a_n}{\displaystyle\int_a^b \frac{dx}{T(b,x,\alpha) f_n(x)}}. \tag{15}$$

Notice that the improvement of Proposition 9 over Proposition 2 is not just to go from Riemann-integrable functions to Lebesgue-integrable functions (which would already be a major improvement), but that we allow the functions to be simply measurable, even if they are not integrable. Also, in Proposition 9 we allow both finite sums and series. Furthermore, we remove the hypothesis $\int \phi = 1$.

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Data Availability
No datasets were generated or analysed during the current study.

Declarations

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Not applicable.

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The authors declare no competing interests.

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