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Explicit Modeling of Multi-Product Customer Orders in a Multi-Period Production Planning Model

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Abstract: In many industries, companies receive customer orders that include multiple products. To simplify the use of optimization models for planning purposes, these orders are broken down, and the quantities of each product are grouped with the same products from other orders to be completed in the same period. Consequently, traditional production planning models enforce minimum demand constraints by product and period rather than by individual orders. An important drawback of this aggregation procedure is that it requires a fixed order fulfillment period, potentially missing opportunities for more efficient resource use through early completion. This paper introduces a novel mathematical formulation that preserves the integrity of customer orders, allowing for early fulfillment when possible. We compare a traditional linear programming model with a new mixed-integer programming approach using a sawmill case study. Although more complex than the traditional model, the proposed formulation reduces costs by approximately 6% by enabling early order completion and offers greater flexibility and control over the production process. This approach leads to better resource utilization and more precise order management, presenting a valuable alternative to conventional production planning models.

Keywords: production planning; customer orders; multi-product orders; mathematical modeling; mixed-integer programming

MSC: 90B30

check for updates

Citation: Palma, C.D.; Vergara, F.P.; Muñoz-Herrera, S. Explicit Modeling of Multi-Product Customer Orders in a Multi-Period Production Planning Model. *Mathematics* **2024**, *12*, 3029. https://doi.org/10.3390/math12193029

Academic Editor: Hu-Chen Liu

Received: 31 July 2024
Revised: 9 September 2024
Accepted: 9 September 2024
Published: 27 September 2024

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1. Introduction

Production planning involves determining the optimal way to utilize available resources and capacities to meet customer demand over a specified time horizon. It encompasses key decisions such as what products to produce, when to produce them, and in what quantities, while considering factors like raw material availability, machine capacities, labor constraints, setup times, and inventory costs, among others [1]. Effective production planning aims to maximize throughput, reduce costs, and ensure on-time delivery by generating feasible production schedules that make efficient use of the manufacturing system's capabilities. Therefore, production planning models are essential tools for optimizing manufacturing operations and supply chain management [2,3].

Optimization models are widely used to address various types of production planning problems, each with distinct objectives, constraints, and time horizons. Karimi et al. [1] discuss the various variants of the single-level lot-sizing problem, one of the most common types of production problems, where all produced items are final and there is a setup cost for production. They present the classical formulation used to include these variants and suggest that further research is needed to develop more efficient solution methods. Missbauer and Uzsoy [2] highlight that production models cover a wide range of managerial levels, problem environments, and time scales, focusing on multi-level problems where products are assembled from other components or products produced in a multistage process. According to [4], the most common variants of the production problem include considerations for backlogs, setup times, overtime capacities, the use of parallel machines, and the multi-site configuration of the production process. Other studies have also considered uncertainty in certain parameters. For example, randomness in demand and production levels are considered in [5], and randomness in demand and fuzziness in different costs in the context of a multi-objective problem are considered in [6]. As noted in [2], there has been scarce focus on the formulation and modeling aspects of the problem. This is also observed in all the previously mentioned variants, which are based on the same mathematical formulation. Whether obtained from market forecasts or customers' orders, product demands in each period over the planning horizon are the main parameters in all production planning models. Many companies face customer multi-product orders with a known due date [7]. Instead of introducing these orders directly into decision models, total product demands per period are used, derived from the delivery dates of various customer orders. Although this approach is practical for resolution purposes, it may introduce limitations and inefficiencies, as orders are forced to be completed within predefined periods, disregarding the potential for more efficient production schedules that allow for earlier order completion.

To address this issue, this paper introduces a novel production planning model that allows for the early fulfillment of customer orders when possible by explicitly maintaining the details of multi-product orders across multiple periods. Unlike traditional models, our approach preserves the identity of each customer order, allowing for more flexible and precise scheduling. This new model is applied to a case study involving a Chilean sawmill, and its performance is compared to that of a traditional production planning model. Through this comparison, we demonstrate the advantages of our proposed model in terms of improved resource utilization, reduced waste, and enhanced scheduling flexibility.

2. Literature Review

The production process involves converting raw materials into final products through multiple transformation steps. In a multiperiod setting described by discrete periods, this requires numerous decisions regarding the acquisition, inventory, and use of raw materials at different times and rates, as well as the production and delivery of products to customers. To support these decisions, production planning models have been successfully applied across a wide range of industries, demonstrating versatility and effectiveness in manufacturing, process industries, and service industries [4]. Production planning problems can be categorized into strategic, tactical, and operational problems [8], although problems involving different decision levels also exist [9,10]. Strategic planning focuses on long-term decisions such as resource and infrastructure investments, supply chain design, and plant location and layout. These decisions aim to establish a competitive position and sustain business growth. Tactical planning addresses resource allocation over a medium-term horizon, involving decisions on material flow, inventory, capacity utilization, and maintenance, to improve cost efficiency and customer satisfaction. Finally, operational planning concentrates on executing production tasks, such as production sequencing and input/output analysis, over a short-term horizon using detailed information to ensure efficient and accurate plan execution [4,11]. For a deeper understanding of the different classifications of production problems and common terminology, see [2,11,12].

While numerous mathematical approaches have been employed to address production planning, this paper focuses on deterministic optimization models at the tactical level. These models are the most common and form the foundation upon which other versions of the problem are built. Linear, integer, and mixed-integer programming are among the most prevalent solution techniques for these models [9,13]. The general structure includes key components such as demand satisfaction, cost or profit objective functions, and the intertemporal relationship between production and inventory decisions. Although models have been classified as capacitated or uncapacitated, with constant or variable demand and involving single or multiple items [11], practical situations typically involve capacitated models with variable demand and multiple items, as considered in this paper.

As described in [14], customers typically order multiple products, specifying desired quantities and delivery dates. Producers, through their sales departments, may negotiate these delivery dates to fit capacity limits, often agreeing on a range of acceptable dates. If an agreement is reached, orders are integrated into various levels of the planning process. We define a customer order as a set of products requested by a customer, distinguishing it from a production or work order, which refers to the company's internal requirement to produce these items. At the tactical level, which this paper addresses, once a period within the agreed delivery range is established, a model determines resource allocation over time and makes production and inventory decisions accordingly. In all cases, multi-product orders are decomposed, and models are fed with product quantities (rather than complete orders). These quantities, combined with products from other customer orders, determine the total amount of each product to be produced in each period [15]. As far as we know, there are no production planning models that preserve the identity of customer orders in their formulation. However, as we will demonstrate later, maintaining this feature can be beneficial in certain aspects.

Various modifications have been made to the traditional model setting. For instance, options for backlogs to meet demand in subsequent periods [11,16–18] or even the loss of sales if demand cannot be met [19,20] are common additions to the demand side of the problem. In [11], the possibility of delivering demand late with a penalty cost, among other variants, is introduced in a basic formulation of a single-item uncapacitated lot-sizing problem with a known demand. In a similar setting, ref. [16] presents a stochastic model with probabilistic demands and returns that allows for unmet demand subject to penalties. A deterministic and stochastic version of a model allowing penalized backlogs is presented in [18], and a model with backlogs and multiple work centers is described in [17]. Beyond backlogs that meet demand after the deadlines, ref. [19] introduces a variant where a fraction of the delayed delivered demand results in a total loss of sales. In [20], no backlogging is allowed, and unmet demand in a period is lost. On the production side, single-level and multi-level models have been used to represent the production of only final products or also the components required to produce the final products, respectively. In single-level settings, models focus on the production of end products without considering the complexity involved in assembling multiple components. In contrast, multi-level models are designed to manage more intricate production processes that require multiple stages of assembly. These models are commonly used in industries such as automotive, electronics, steel-making, and textiles, where products consist of various components [21]. Additionally, the option to split customer orders across multiple plants to balance production loads [22] and the incorporation of environmental concerns have been explored through multi-objective models [23,24]. For example, in [23], a score for each product is calculated based on environmental criteria such as biodegradability, energy consumption, and recyclability, alongside costs, resulting in a set of Pareto-optimal solutions. Similarly, in [24], the environmental impact of the production and the social benefits, measured in terms of job creation, are combined with the traditional cost minimization objective function.

The sawmill industry has benefited from the application of production models. Sawmill production involves converting logs into various lumber products through a series of cutting and processing stages. This process is complex and requires careful planning to maximize resource utilization, and optimization models are extensively used to address these challenges. These models help in making critical decisions regarding log allocation, cutting patterns, inventory management, and production scheduling. Typical constraints in these models include the availability of raw materials (logs), production capacities, and lumber demand requirements. Unlike other industries, a wide range of different objectives have been proposed for sawmill problems [25], and multi-objective models have been developed to balance conflicting goals [26]. The presence of uncertainties, mainly in the yield of logs and in the demand, has also been considered in the context of sawmill operations [27–29].

This paper uses the context of a sawmill to evaluate a novel mixed-integer programming formulation for the production planning problem, where the identity of customer orders is preserved. The methodology section presents this new formulation and describes the case study used to compare it with the traditional formulation. Following this, the presentation and discussion of results demonstrate the model's performance, highlighting its advantages and drawbacks over traditional approaches. Finally, the conclusion summarizes our findings and suggests potential areas for future research.

3. Materials and Methods

We compared a traditional production planning model with our new planning model, which preserves the details of production orders, through a practical application. This comparison was based on a set of three simulated instances reflecting the operation of a sawmill. The instances were generated based on the operations of one of the largest sawmills in Chile, located in the Biobio region and owned by CMPC Forest Company (Santiago, Chile). This sawmill has been in operation since 2007 and has a production capacity of 315,000 cubic meters of radiata pine-sawn timber, serving both national and international markets. Like many sawmills, it faces the ongoing challenge of meeting increasingly complex customer demands while managing a limited supply of logs.

Next, we describe the decision problem in detail, the two optimization models compared, and the data used for our case study.

3.1. Problem Description

The sawmill industry operates at the intersection of natural resource management, manufacturing efficiency, and customer satisfaction. The core of the sawmill operation lies in the judicious use of logs sourced from various timber species. Each log possesses unique attributes, including dimension (diameter and length), curvature, and wood quality, among others, which must be carefully considered to meet the diverse requirements of the lumber market. The sawmill planning process involves the meticulous selection of the cutting pattern to be applied to each log. A cutting pattern is a scheme of cuts used to obtain a set of rectangular cross-section lumber from logs. The type and number of lumber pieces obtained when applying a cutting pattern depend on the size of the logs. It should be noted that a lumber piece can be obtained from different types of logs and using different cutting patterns. Although there is an optimal cutting pattern for each log size individually (for example, one that minimizes waste), suboptimal cutting patterns must sometimes be applied to meet customer demands. A more detailed description of the problem can be found in [25].

As in other industries, sawmill customer demand usually consists of orders composed of different types of products [15,30]. When the sawmill receives a production order, a date for order fulfillment is defined based on the sawmill's workload. Let us assume that an order $k \in K$ is accepted and scheduled to be completed in period $t \in T$, and that a_{pk} is the volume of product $p \in P$ required in order k. Using a_{pk} and the binary parameter o_{kt}, which equals 1 if order k is scheduled for period t and 0 otherwise, production orders can be translated into an overall demand d_{pt}, calculated as follows:

$$d_{pt} = \sum_{k \in K | O_{kt} = 1} a_{pk} \quad \forall p, t \tag{1}$$

Sawmill operations are planned to satisfy this multi-product demand over time while meeting resource availability constraints, including log supply and processing capacity [25].

3.2. Traditional Production Planning Model

Although production models may incorporate features specific to each application, they all allocate resources to meet product demands. Objective functions can vary widely [25] or combine different criteria [26]. In our case, we focus on cost minimization. To better understand the behavior of our new problem formulation, we modeled a simplified version of the problem. This approach allows us to concentrate on the general structure of the new formulation and its implications. For our case study, we formulated the following model:

Decision variables:

- x_{ijt} = volume of log type i processed with sawing pattern j in period t (m^3).
- y_{pt} = volume of product p produced in period t (m^3).
- z_{pt} = volume of product p held as inventory in period t (m^3).

Parameters/coefficients:

- c_i^{log} = cost of log type i (USD/m^3).
- c_i^{proc} = processing cost of log type i (USD/m^3)
- c_p^{inv} = inventory cost of product p (USD/m^3)
- r_{ijp} = volume of product p obtained if a log i is processed with sawing pattern j (m^3).
- d_{pt} = volume demanded of product p in period t (m^3).
- pt_i = processing time of a log type i (h/m^3).
- q_t = available processing time in period t (h).

Objective function (cost minimization):

$$Min \sum_{i\in I}\sum_{j\in J}\sum_{t\in T}\left(c_i^{log}+c_i^{pro}\right)x_{ijt}+\sum_{p\in P}\sum_{t\in T}c_p^{inv}y_{pt} \tag{2}$$

Subject to the following set of constraints:

$$\sum_{i\in I}\sum_{j\in J}r_{ijp}x_{ijt}-y_{pt}\geq 0 \quad \forall p,t \tag{3}$$

$$y_{pt}+z_{p(t-1)}-z_{pt}=d_{pt} \quad \forall p,t \tag{4}$$

$$\sum_{i\in I}\sum_{j\in J}pt_i x_{ijt}-q_t\leq 0 \quad \forall t \tag{5}$$

All variables are non-negative. Constraints (3) transform logs into lumber boards, constraints (4) ensure demand fulfillment by relating production and inventory decisions, and constraints (5) limit the processing time to its maximum capacity.

3.3. Proposed Production Planning Model

For simplicity, traditional models omit any reference to production orders, combining them using Equation (1) and scheduling production at the product level by period. In contrast, the proposed formulation maintains the identity of production orders, which has certain advantages that will be discussed later.

Decision variables:

- x_{ijt} = volume of log type i processed with sawing pattern j in period t (m^3).
- v_{pkt} = volume of product p produced in period t to fulfill order k (m^3).
- u_{pkt} = fraction of the demand for product p in order k held as inventory in period t.
- w_{kt} = binary variable that values 1 if order k is completed in period t, 0 otherwise.

Parameters/coefficients:

We add the following data to the ones described for the traditional model.

- o_{kt} = binary parameter equals 1 if order k must be finished by period t, 0 otherwise.
- a_{pk} = volume of product p required in order k (m^3).
- n_k = number of different products in order k.

Objective function (cost minimization):

$$Min \sum_{i\in I}\sum_{j\in J}\sum_{t\in T}\left(c_i^{log}+c_i^{pro}\right)x_{ijt}+\sum_{p\in P}\sum_{k\in K}\sum_{t\in T}c_p^{inv}\left(a_{pk}u_{pkt}-w_{kt}\right) \quad (6)$$

Constraints (5) are also part of this formulation, but constraints (3) and (4) are replaced with the following set of constraints:

$$\sum_{i\in I}\sum_{j\in J}r_{ijp}x_{ijt}-\sum_{k\in K}v_{pkt}\geq 0 \quad \forall p,t \quad (7)$$

$$u_{pkt}-\frac{1}{a_{pk}}\sum_{s\in T|s\leq t}v_{pks}+\sum_{r\in T|r<t}w_{kr}=0 \quad \forall p,k,t \quad (8)$$

$$\sum_{s\in T|s\leq t}w_{ks}-1=0 \quad \forall k,t|o_{kt}=1 \quad (9)$$

$$\sum_{p\in P}u_{pkt}-n_k w_{kt}\geq 0 \quad \forall k,t \quad (10)$$

Decision variables x_{ijt} and v_{pkt} are non-negative, $0 \leq u_{pkt} \leq 1$, $\forall p, k, t$ and $w_{kt} \in \{0,1\}$, $\forall k,t$. The objective function (Equation (6)) differs from the traditional one in how inventory costs are calculated. In this formulation, inventory is determined by the fraction u_{pkt} of the demand a_{pk} that has been produced up to a period t. However, in the period when u_{pkt} becomes 1 (i.e., the order is completed), the inventory cost is not considered because the order is delivered to the customer. During this period, when $w_{kt}=1$, the inventory cost of the completed order is subtracted. Constraints (7) transform logs into lumber boards, which are allocated to different orders. Constraints (8) calculate the fraction of demand produced up to a certain period and force this fraction to take a value of 0 from the period in which the order is completed. Constraints (9) require each order to be finished any time before its deadline, identified as the time when $o_{kt}=1$. Finally, constraints (10) identify the actual completion period of each order.

It is worth noting that the proposed formulation behaves like the traditional one when the decision on the completion time for each order is constrained to be its due date, that is, when w_{kt} equals o_{kt}, for all k and t. Under this condition, all customer orders k are completed exactly as originally scheduled with the clients, preventing any possibility of early completion as permitted in the proposed formulation. This means that the amount of product p, denoted as a_{pk} in the proposed model, for all orders k set to be completed in period t, will only become available in period t, and not before. Consequently, the amount of product in each period is determined by Equation (1), and the actual production is identical in both formulations. In this scenario, while the proposed model still offers the advantage of tracking production progress by customer order, it does not demonstrate benefits in terms of more efficient resource use.

3.4. Data and Computational Experiments

Both models were compared using three different datasets from a Chilean sawmill to assess the impact of instance size on their performance. Although there is a wide variety of sawing patterns available, most sawmills use a limited set of patterns to fulfill specific production orders. Therefore, we considered 15 sawing patterns across all datasets. Table 1 shows the sizes of the instances on which the models were tested.

Table 1. Description of the instance used to compare models.

Instance Name	Number of			
	Logs	**Periods**	**Orders**	**Products**
Small (S)	50	5	10	20
Medium (M)	100	10	20	40
Large (L)	150	15	30	60

4. Discussion of Model Applications

In this section, we present and discuss the results of our study comparing the performance of a traditional production model with our proposed approach. We focus on the computational behavior of the models, specifically examining model size, solution times, and the nature of the solutions provided.

4.1. Computational Behavior

The models were implemented using IBM ILOG CPLEX Optimization Studio v22.1 [31] and executed on an Intel Core i5 processor with a 1.60 GHz clock speed and 8 GB of RAM (Intel Corp., Santa Clara, CA, USA). The resulting model sizes and solution times are presented in Table 2.

Table 2. The number of variables (binaries in parenthesis), constraints, and solution times for each model.

Instance Name	Traditional			Proposed			
	Variables (Binaries)	**Constraints**	**Time (s)**	**Variables (Binaries)**	**Constraints**	**Time (s)**	**Warm Start (s)**
Small (S)	3950 (0)	205	0.3	5750 (50)	1165	1.1	1.7
Medium (M)	15,800 (0)	810	0.8	31,000 (200)	8630	24.4	38.5
Large (L)	35,550 (0)	1815	1.8	87,750 (450)	28,395	2595.6	542.3

Due to the significant increase in solution times for the proposed formulation, we employed a warm start by providing initial values for certain variables. A warm start, also known as MIP start, involves providing an initial solution as a hint to the solver to expedite the optimization process. This technique can significantly reduce computation time by guiding the solver towards promising regions of the solution space. The effectiveness of warm-start techniques has been well documented, particularly in complex and large-scale optimization problems [32,33]. In our case, an effective initial solution for the binary variables was determined based on the periods of the order deadlines; therefore, we set $w_{ky}^{ini} = o_{kt}$ for all k and t. A relative gap of 5% was defined for all MIP models.

Examining the size of the models, the proposed formulation consistently required a larger number of variables and constraints compared to the traditional approach. This was expected as the proposed model aims to maintain the identity of production orders, resulting in a more detailed representation of the production process. Although the increase in the number of decision variables was expected, the main concern of the proposed formulation is the introduction of binary variables. Despite their relatively lower count even in the largest instance, the presence of binary variables can significantly complicate the solution process [34], as observed in the prolonged solution times.

The traditional model demonstrated relatively stable solution times across all dataset sizes, with modest increases as the size of the dataset grows. Conversely, the proposed model exhibited a more pronounced escalation in solution times with larger datasets. This was especially evident in the largest instance (L), where the solution time for the proposed model exceeded 2500 s. Such extended solution times suggest potential scalability issues with the proposed model, particularly when dealing with extensive datasets. However, implementing a simple yet effective warm start for the binary variables significantly reduced the solution time for the larger instance. While the solution times increased for smaller instances, as noted in other applications [32], the warm start approach appears to be recommended for medium to large instances.

The proposed model exhibited a notable increase in the number of constraints as the size of the problem instances grew. This rise in constraints directly stems from the model's effort to preserve the identity of production orders. Specifically, calculating the progress completion of orders and its correlation with a new decision variable for the completion period entails the incorporation of numerous additional constraints. Although the number of decision variables and constraints directly affect the solution process, it is worth noting that the solution time of MIP models is influenced not only by their size, defined by the number of binary variables and constraints, but also by the model's complexity, which is determined by the choice of dataset, constraints formulation, and objective function. The dataset employed directly shapes the structure of the objective function, impacting the coefficients within it as well as the coefficients and bounds of constraints [35,36]. These factors collectively contribute to the computational burden and ultimately affect the solution time of the models.

Therefore, while the inclusion of a larger number of constraints in the proposed model may contribute to its effectiveness in capturing the intricacies of production processes, it also poses challenges in terms of computational scalability and solution time, necessitating careful consideration and potentially requiring advanced algorithmic techniques to ensure efficient solution methods.

4.2. Comparison of Model Decisions

In this section, we conduct a comparative analysis of model decisions with a particular emphasis on how production and inventory decisions evolve in both models. Our goal is to highlight the advantage of the proposed model in expediting order completion. It is noteworthy that the proposed model offers the flexibility to fulfill demands ahead of schedule, unlike the traditional model. Moreover, when this flexibility is removed from the proposed formulation, i.e., when w_{kt} equals o_{kt}, for all k and t, both models exhibit the same behavior. By preserving the identity of production orders, the proposed model can strategically advance the production of various products within an order to expedite its completion, resulting in cost savings. The total costs are presented in Table 3, showing a 5.8% reduction for instances M and L.

Table 3. Total plan cost of models run.

Instance Name	Total Cost (USD)		
	Traditional	**Proposed**	**Savings (%)**
Small (S)	3,497,891	3,488,409	0.3
Medium (M)	8,519,665	8,026,729	5.8
Large (L)	15,777,107	14,867,580	5.8

As the proposed model allows for advanced production scheduling, cost savings arise not only from reduced inventory expenses but also from optimized resource utilization (specifically, logs and processing). Figure 1 illustrates the cost trends for the three instance sizes (S, M, and L) across both models, showcasing reductions in all cost components with the adoption of the proposed model. It is worth noting that in the case of the S instance, there was a slight increase in inventory costs.

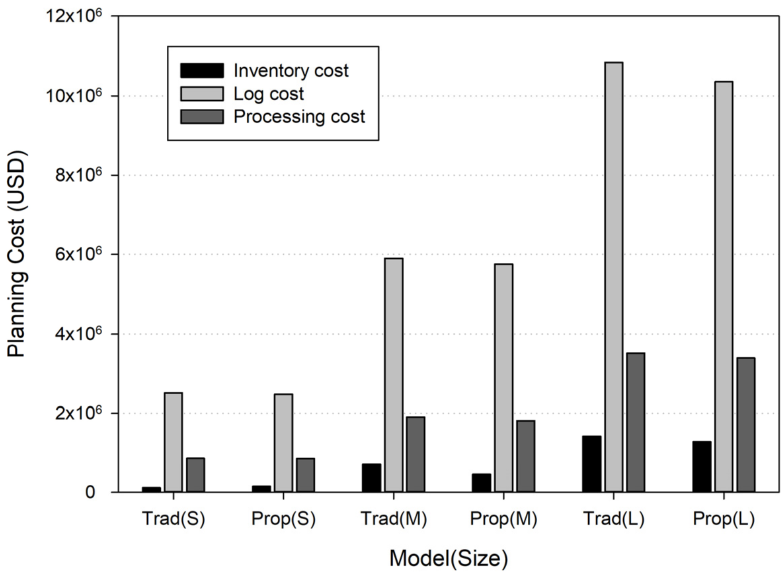

Figure 1. The cost of the different components decreased with the proposed model.

Figure 2 illustrates the quantity of logs required and the resulting residues produced. The volume of logs that were not converted to lumber was considered residues, represented by the slacks in constraints 3 and 7. When producing lumber pieces, logs of varying characteristics are cut using the most suitable sawing pattern, inevitably resulting in some wastage [25]. Given the flexibility afforded by the proposed approach in scheduling production order completion, it can more effectively match logs, sawing patterns, and products, thereby enhancing the utilization of raw materials. Residues were reduced by 16%, 49%, and 35% for the three instance sizes, respectively, resulting in a decreased demand for logs. Specifically, log usage decreased by 2.4%, 3.2%, and 5.2% for each instance size, respectively. This reduction in log usage is particularly significant for the sawmill industry, as logs represent around 70% of the total production cost and their supply may become an issue [37].

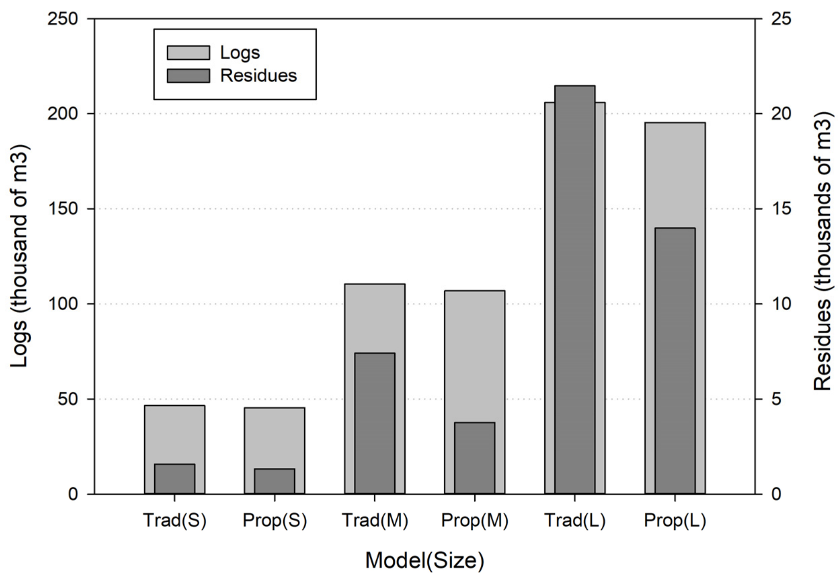

Figure 2. A significant reduction in residues and logs used was observed with the proposed model.

One of the characteristics of the proposed approach is that it explicitly represents the progress of customer order completions. Figure 3 shows, as an example, the detailed work progress of customer order 5. This order contains ten products, whose production was filled at different rates during the periods (variable u_{pkt}). In this example, the customer order is finished in period 4 (i.e., $w_{54} = 1$), where all fraction variables u_{pkt} become one. Maintaining detailed information on the progress of the production of customer orders offers some advantages. It enables better planning and prioritization by identifying and expediting critical products for important or urgent customers, leading to enhanced customer satisfaction. Detailed tracking improves communication with customers by providing precise updates on order statuses, fostering transparency and trust. This detail also allows for dynamic adjustments to production in response to changes in demand or customer priorities, increasing flexibility. Additionally, detailed information facilitates in-depth analysis for continuous improvement, identifying bottlenecks and optimization opportunities.

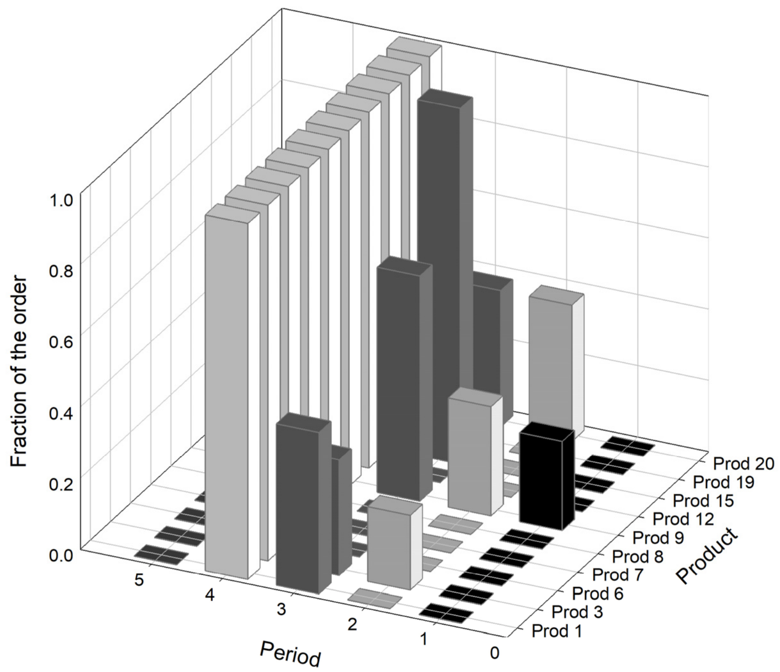

Figure 3. Variable u_{pkt} in the proposed formulation models the progress of customer order production as a fraction, allowing for tracking each product's advancement over time. The figure shows a customer order fully completed by period 4.

5. Conclusions and Future Work

This paper presents a novel mixed-integer formulation of an optimization model for a multi-period multi-product cost minimization production planning problem that explicitly maintains the identity of customer orders within the model. Traditional optimization models combine different customer orders and determine the overall amount of each product to produce over time, without considering the specific order to which the product belongs. This conventional approach requires orders to have a fixed completion period. In contrast, our formulation maintains the identity of customer orders, providing detailed tracking of each order's production progress and allowing orders to be completed ahead of their deadlines, thus optimizing resource use. The key innovations introduced in our formulation include a production decision variable specific to each customer order, defining inventory decisions as a fraction of demand, and incorporating binary variables to identify the period in which an order is completed. Additional constraints were introduced to integrate these new variables. When tested in a sawmill case study, our formulation showed nearly a 6% reduction in total cost compared to the traditional model, mainly due to completing orders earlier and making more efficient use of resources. Additionally, the variables introduced in our formulation provide enhanced control over the production process and the ability to prioritize customers and critical products. This also facilitates precise updates on order statuses, leading to improved customer satisfaction.

However, the proposed formulation has certain limitations. It is more complex than traditional models, featuring a significant increase in the amount of continuous variables and constraints. Additionally, the inclusion of binary variables makes the model harder to solve. For the largest instance evaluated, the number of variables increased from roughly 35,500 to 87,700 (including 450 binary variables), and the number of constraints rose from 1800 to over 28,000. As a result, solution time increased from 2 to 542 s, highlighting the added computational effort required for the more complex model. Additionally, the proposed formulation represents a basic production model. This simplification was intentional to introduce the new modeling structure more clearly, but it may limit the model's application in more complex, real-world scenarios.

Future Work

Future work should focus on two main areas. First, given the basic structure of the order-centric production model presented in this study, exploring various problem variants would be valuable. This could involve the inclusion of backorders, which would introduce flexibility in meeting demand over multiple periods and extending the model to a multi-level production setting. A multi-level framework would allow for the integration of more complex production processes where different components are assembled, thus broadening the model's applicability to industries beyond sawmills.

Second, due to the increased complexity and longer solution times associated with the proposed model, future research should focus on improving the model's scalability. This could be achieved by investigating stronger mathematical formulations and leveraging advanced optimization techniques. Such improvements would help address the computational challenges posed by large-scale instances of the proposed model and its potential variants.

Author Contributions: Conceptualization, C.D.P. and F.P.V.; data curation, C.D.P. and F.P.V.; formal analysis, C.D.P.; investigation, C.D.P.; methodology, C.D.P.; resources, C.D.P.; software, C.D.P.; supervision, C.D.P.; validation, C.D.P. and F.P.V.; visualization, C.D.P.; writing—original draft, C.D.P., F.P.V. and S.M.-H.; writing—review and editing, C.D.P., F.P.V. and S.M.-H. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: All data supporting reported results are openly available from Zenodo at https://doi.org/10.5281/zenodo.13498389 (accessed on 31 July 2024).

Acknowledgments: The authors thank the editor for her support and assistance throughout the review process, and the two reviewers for their constructive suggestions. Their feedback has significantly contributed to the improvement of our manuscript.

Conflicts of Interest: The authors declare no conflicts of interest.

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