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Article

On the Inverse Degree Polynomial

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Abstract: Using the symmetry property of the inverse degree index, in this paper, we obtain several mathematical relations of the inverse degree polynomial, and we show that some properties of graphs, such as the cardinality of the set of vertices and edges, or the cyclomatic number, can be deduced from their inverse degree polynomials.

Keywords: Inverse degree index; inverse degree polynomial; polynomials; topological indices

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1. Introduction

The interest in topological indices lies in the fact that they synthesize some of the fundamental properties of a molecule into a single value, and therefore, find many applications in chemistry. With this in mind, several topological indices have been studied so far; we note the seminal work by Wiener [1] in which he used the distances of a chemical graph in order to model the properties of alkanes. In particular, the inverse degree index $ID(G)$ of a graph G is defined by

$$ID(G) := \sum_{u \in V(G)} \frac{1}{d_u} = \sum_{uv \in E(G)} \left(\frac{1}{d_u^2} + \frac{1}{d_v^2} \right),$$

where $d_u = \deg_G u$ denotes the degree of the vertex u in G.

The first time that the inverse degree of a graph attracted attention was due to numerous conjectures that were generated by the computer program Graffiti [2]. Since then, in many works by several authors (e.g., [3–7]), the relationship between other graph invariants, such as diameter, edge-connectivity, matching number and Wiener index, have been studied.

The study of polynomials in graphs is a very useful subject from a theoretical and practical point of view and is of growing interest. In particular, in [8], there is an extensive development of the domination polynomial, while in [9], we find a study of the roots of independent polynomials. In both cases, the fundamental idea was to have a characterization of the graphs from the polynomials.

In the work by Baig et al. [10], the authors studies Omega, Sadhana, and PI counting polynomials and computed topological indices associated with them. These polynomials are used to predict several physico-chemical properties of certain chemical compounds. The same three polynomials were computed in a more recent work by Imran et al. [11]. Here the authors computed polynomials for mesh-derived networks.

We want to recall that polynomials in graphs have been widely used to study the structural properties of certain graph families. Several graph parameters have been used to define a graph polynomial, for instance, differential number, the parameters associated to matching, independent and domination sets, chromatic numbers and many others (see, e.g., [12] and the references therein). In recent years there have been many works on graph polynomials associated with different topological indices (see, e.g., [13–17]).

In [13], Shuxian defined the Zagreb polynomial of a graph G as

$$M_1^*(G,x) := \sum_{u \in V(G)} d_u x^{d_u}.$$

In [14] the harmonic polynomial of a graph G is defined as

$$H(G,x) := \sum_{uv \in E(G)} x^{d_u+d_v-1}.$$

The harmonic polynomials of the line graphs were studied in [17]. The inverse degree polynomial is studied to understand better the inverse degree topological index. In [15], this polynomial was used in order to obtain bounds of the harmonic index of the main products of graphs; in order to do that, a main tool was the inverse degree polynomial (or ID polynomial) of a graph G, defined as

$$ID(G,x) = \sum_{u \in V(G)} x^{d_u-1}.$$

Thus, we have $\int_0^1 ID(G,x)\,dx = ID(G)$. So, one can expect to obtain information on the inverse degree index from the properties of the inverse degree polynomial. Note that,

$$x(xID(G,x))' = M_1^*(G,x).$$

2. Some Preliminaries

The next interesting result appears in [15].

Proposition 1. *If G is a graph with n vertices, and k of them are pendant vertices, then:*

- $ID^{(j)}(G,x) \geqslant 0$ *for every* $j \geqslant 0$ *and* $x \in [0,\infty)$,
- $ID(G,x) > 0$ *on* $(0,\infty)$,
- $ID(G,x)$ *is strictly increasing on* $[0,\infty)$ *if and only if G is not isomorphic to an union of path graphs* P_2,
- $ID(G,x)$ *is strictly convex on* $[0,\infty)$ *if and only if G is not isomorphic to an union of path graphs and/or cycle graphs, and*
- $k = ID(G,0) \leqslant ID(G,x) \leqslant ID(G,1) = n$ *for every* $x \in [0,1]$.

The following results are direct.

Proposition 2. *If G is a k-regular graph with n vertices, then* $ID(G,x) = nx^{k-1}$.

The following result computes the ID polynomial of: K_n (the complete graph with n vertices), C_n (the cycle with $n \geqslant 3$ vertices), Q_n (the n-dimensional hypercube), K_{n_1,n_2} (the complete bipartite graph with n_1+n_2 vertices), S_n (the star graph with n vertices), P_n (the path graph with n vertices), W_n (the wheel graph with $n \geq 4$ vertices), and S_{n_1,n_2} (the double star graph with n_1+n_2+2 vertices).

Proposition 3. *We have*

$$\begin{aligned} ID(K_n,x) &= nx^{n-2}, & ID(C_n,x) &= nx, \\ ID(Q_n,x) &= 2^n x^{n-1}, & ID(K_{n_1,n_2},x) &= n_1 x^{n_2-1} + n_2 x^{n_1-1}, \\ ID(S_n,x) &= x^{n-2} + n - 1, & ID(P_n,x) &= (n-2)x + 2, \\ ID(W_n,x) &= x^{n-2} + (n-1)x^2, & ID(S_{n_1,n_2},x) &= x^{n_1} + x^{n_2} + n_1 + n_2. \end{aligned}$$

The forgotten topological index (or F-index) is defined as

$$F(G) = \sum_{uv \in E(G)} (d_u^2 + d_v^2) = \sum_{u \in V(G)} d_u^3.$$

3. Main Results

The next result allow us to obtain information about the graph using the ID polynomial.

Proposition 4. *If G is a graph with n vertices and m edges, then we have*

- $ID(G,1) = n$,
- $ID'(G,1) = 2m - n$,
- $ID''(G,1) = M_1(G) - 6m + 2n$, *and*
- $ID'''(G,1) = F(G) - 6M_1(G) + 22m - 6n$.

Proof. The equality $ID(G,1) = n$ is a consequence of Proposition 1. By the handshaking Lemma, we have

$$ID'(G,x) = \sum_{u \in V(G)} (d_u - 1)\, x^{d_u - 2},$$
$$ID'(G,1) = \sum_{u \in V(G)} d_u - \sum_{u \in V(G)} 1 = 2m - n,$$

and

$$\begin{aligned} ID''(G,x) &= \sum_{u \in V(G)} (d_u - 1)(d_u - 2)\, x^{d_u - 3}, \\ ID''(G,1) &= \sum_{u \in V(G)} d_u^2 - 3 \sum_{u \in V(G)} d_u + 2 \sum_{u \in V(G)} 1 \\ &= M_1(G) - 6m + 2n, \\ ID'''(G,x) &= \sum_{u \in V(G)} (d_u - 1)(d_u - 2)(d_u - 3)\, x^{d_u - 4}, \\ ID'''(G,1) &= \sum_{u \in V(G)} d_u^3 - 6 \sum_{u \in V(G)} d_u^2 + 11 \sum_{u \in V(G)} d_u - 6 \sum_{u \in V(G)} 1 \\ &= F(G) - 6M_1(G) + 22m - 6n. \end{aligned}$$

□

Corollary 1. *If G is a graph with n vertices and m edges, then we have*

- $n = ID(G,1)$,
- $m = \frac{1}{2}\big(ID'(G,1) + ID(G,1)\big)$,
- $M_1(G) = ID''(G,1) + 3ID'(G,1) + ID(G,1)$, *and*
- $F(G) = ID'''(G,1) + 6ID''(G,1) + 7ID'(G,1) + ID(G,1)$.

Proposition 2 shows that any two k-regular graphs with the same cardinality of vertices, have the same ID polynomial. The following question is very natural: How many graphs can be characterized by their ID polynomials? Although this is a very difficult question, Corollary 1 provides a partial answer: Graphs with different cardinality of vertices or edges have different ID polynomials. This property allows to conclude the following.

Corollary 2. *If* Γ *is a proper subgraph of the graph G, then* $ID(\Gamma, x) \neq ID(G,x)$.

Also, we can obtain information about the cycles in the graph by using the ID polynomial. The *cyclomatic number* of a connected graph with n vertices and m edges is defined as $\gamma(G) = m - n + 1$.

Proposition 5. *If G is a connected graph, then its cyclomatic number is*

$$\gamma(G) = \frac{1}{2}\big(ID'(G,1) - ID(G,1)\big) + 1.$$

In particular, if $ID(G,1) = ID'(G,1) + 2$, then G is a tree.

Proof. Corollary 1 gives $n = ID(G,1)$ and $m = \frac{1}{2}\big(ID'(G,1) + ID(G,1)\big)$. Hence,

$$\gamma(G) = m - n + 1 = \frac{1}{2}\big(ID'(G,1) + ID(G,1)\big) - ID(G,1) + 1 = \frac{1}{2}\big(ID'(G,1) - ID(G,1)\big) + 1.$$

If $ID(G,1) = ID'(G,1) + 2$, then $\gamma(G) = 0$ and G is a tree. □

Furthermore, Proposition 6 and Theorem 7 show that two graphs with the same ID polynomial have to be similar, in some sense.

If k is any positive integer, we consider

$$R_k(x) := (x-1)(x-2)\cdots(x-k) = x^k + \sum_{j=0}^{k-1} a_{k,j} x^j.$$

Vieta's formulas give

$$a_{k,k-j} = (-1)^j \sum_{1 \leqslant i_1 < i_2 < \cdots < i_j \leqslant k} i_1 i_2 \cdots i_j.$$

Note that $a_{k,0} = (-1)^k k!$ and $a_{k,k-1} = -\frac{1}{2}\,k(k+1)$.

The following result generalizes Proposition 4.

Proposition 6. *If G is a graph and k is a positive integer, then*

$$ID^{(k)}(G,1) = M_1^k(G) + \sum_{j=0}^{k-1} a_{k,j} M_1^j(G).$$

Proof. We have

$$\begin{aligned}
ID'(G,x) &= \sum_{u \in V(G)} (d_u - 1)\, x^{d_u - 2}, \\
ID''(G,x) &= \sum_{u \in V(G)} (d_u - 1)(d_u - 2)\, x^{d_u - 3}, \\
ID^{(k)}(G,x) &= \sum_{u \in V(G)} (d_u - 1)(d_u - 2)\cdots(d_u - k)\, x^{d_u - k - 1} \\
&= \sum_{u \in V(G)} R_k(d_u)\, x^{d_u - k - 1}, \\
ID^{(k)}(G,1) &= \sum_{u \in V(G)} R_k(d_u) = \sum_{u \in V(G)} d_u^k + \sum_{j=0}^{k-1} \sum_{u \in V(G)} a_{k,j} d_u^j \\
&= M_1^k(G) + \sum_{j=0}^{k-1} a_{k,j} M_1^j(G).
\end{aligned}$$

□

The next result gives bounds of the ID index in terms of the values of the ID polynomial at the points 0, 1/2 and 1.

Proposition 7. *For any graph G, we have the inequalities*

$$ID(G,1/2) \leqslant ID(G) \leqslant \frac{ID(G,0)+ID(G,1)}{2},$$

and the equality in each inequality is attained if and only if G is isomorphic to an union of path graphs.

Proof. Hermite–Hadamard's inequality gives

$$g(1/2) \leqslant \int_0^1 g(x)\,dx \leqslant \frac{g(0)+g(1)}{2},$$

for every convex function g on $[0,1]$, and both inequalities are strict when g is strictly convex.

If G is not isomorphic to an union of path graphs and/or cycle graphs, then Proposition 1 gives that $ID(G,x)$ is a strictly convex function. So, applying Hermite-Hadamard's inequality to the function $g(x) = ID(G,x)$, we obtain

$$ID(G,1/2) < \int_0^1 ID(G,x)\,dx = ID(G) < \frac{ID(G,0)+ID(G,1)}{2}.$$

If G is isomorphic to an union of path graphs and/or cycle graphs, then the ID polynomial of G has degree at most 1, and so, both inequalities are attained. □

Given a graph G and a vertex $v \in V(G)$, we denote by $N(v)$ the set of neighbors of v. We say that v is *dominant* if $N(v) = V(G) \setminus \{v\}$.

Given a polynomial $P(x)$, let us denote by $\operatorname{Deg} P(x)$ the degree of $P(x)$, and by $\operatorname{Deg}_{\min} P(x)$ the minimum degree of the monomials with non-zero coefficients of $P(x)$.

Proposition 8. *Let G be a graph with n vertices, maximum degree Δ and minimum degree δ. Then:*

- $\operatorname{Deg} ID(G,x) = \Delta - 1$.
- $\operatorname{Deg}_{\min} ID(G,x) = \delta - 1$.
- *$x = 0$ is a zero of $ID(G,x)$ if and only if $\delta > 1$.*
- *If $\delta > 1$, then $x = 0$ is a zero of $ID(G,x)$ with multiplicity $\delta - 1$.*
- $\operatorname{Deg} ID(G,x) \leqslant n-2$, *and the equality is attained if and only if there exists at least a dominant vertex in G.*
- *If Γ is a subgraph of G, then* $\operatorname{Deg} ID(\Gamma,x) \leqslant \operatorname{Deg} ID(G,x)$ *and* $\operatorname{Deg}_{\min} ID(\Gamma,x) \leqslant \operatorname{Deg}_{\min} ID(G,x)$.

Proof. The two first items are a direct consequence of the definition of $ID(G,x)$.

The third and fourth items follow from the second one.

We have $\operatorname{Deg} ID(G,x) = \Delta - 1 \leqslant n-2$, and the equality holds if and only if $\Delta = n-1$, i.e., there exists at least a dominant vertex in G.

The last statement follows from the two first items. □

Theorem 1. *Let G be a graph with minimum degree δ. The point $x = 0$ is the unique zero of the ID polynomial of G if and only if $\delta > 1$ and G is regular.*

Proof. If $\delta > 1$ and G is regular, then Proposition 2 gives that $x = 0$ is the unique zero of $ID(G,x)$.

Assume now that $x = 0$ is the unique zero of the ID polynomial of the graph G. Thus, $ID(G,x) = ax^{b-1}$ for some positive integers a, b, and $d_u = b$ for every $u \in V(G)$, and so, G is regular. Also, Proposition 8 gives $\delta > 1$. □

The next result allows to obtain information about the connectedness and diameter of a graph G in terms of the degree of its ID polynomial.

Theorem 2. *Let G be a graph with n vertices. If $\operatorname{Deg} ID(G,x) = n-2$, then $\operatorname{diam} G \leqslant 2$ and G is a connected graph.*

Proof. Since $\operatorname{Deg} ID(G,x) = n-2$, Proposition 8 gives that there exists at least a dominant vertex v in G. Hence, $d_G(u,v) = 1$ for every $u \in V(G) \setminus \{v\}$. If $u_1, u_2 \in V(G) \setminus \{v\}$, then $d_G(u_1,u_2) \leqslant d_G(u_1,v) + d_G(v,u_2) = 2$. Hence, $\operatorname{diam} G \leqslant 2$ and, consequently, G is a connected graph. □

The following classical result gives an asymptotically sharp upper bound for the diameter of a connected graph (see [18] [Theorem 1]). Recall that, if t is any real number, then $\lfloor t \rfloor$ denotes the lower integer part of t, i.e., the largest integer less than or equal to t.

Theorem 3 (Erdös, Pach, Pollack, and Tuza)**.** *Let G be a connected graph with n vertices and minimum degree $\delta \geq 2$. Then*

$$\operatorname{diam} G \leq \left\lfloor \frac{3n}{\delta+1} \right\rfloor - 1.$$

The two next results in [19] provide better estimations for the diameter of a connected graph.

Theorem 4. *If G is a connected graph with n vertices and minimum degree δ, then $\operatorname{diam} G \leq n-1$ if $\delta = 1$, and*

$$\operatorname{diam} G \leq \max\left\{2, \left\lfloor \frac{3n-4}{\delta+1} \right\rfloor - 1\right\}$$

for every $\delta \geq 2$.

Theorem 5. *Let n be the number of vertices of the connected graph G and with a minimum degree equal to δ.*

i. *If $\delta \geq \dfrac{(n-i)}{2}$, then $\operatorname{diam} G \leq i+1$, $\forall i = 1, \ldots, 5$.*
ii. *If $\delta \geq \dfrac{(n-2)}{3}$, then $\operatorname{diam} G \leq 5$.*
iii. *If $\delta \geq \dfrac{(n-3)}{3}$, then $\operatorname{diam} G \leq 6$.*

These two results in [19] have the following consequences for general graphs.

Proposition 9. *If G is a graph with n vertices, minimum degree δ and at least $r \geqslant 1$ connected components, and Γ is a connected component of G, then:*

i. *If $\delta = 1$ we have $\operatorname{diam} \Gamma \leq n-1-(r-1)(\delta+1)$.*
ii. *If $\delta \geq 2$, then*

$$\operatorname{diam} \Gamma \leq \max\left\{2, \left\lfloor \frac{3n-4}{\delta+1} \right\rfloor - 3r + 2\right\}.$$

Proof. Each connected component of G has at least $\delta+1$ vertices. Since G is a graph with at least $r \geqslant 1$ connected components, G has at least $r(\delta+1)$ vertices, i.e., $r(\delta+1) \leqslant n$. Also, any connected component of G has at most $n-(r-1)(\delta+1)$ vertices.

If $\delta = 1$, then

$$\operatorname{diam} \Gamma \leq |V(\Gamma)| - 1 \leq n-1-(r-1)(\delta+1).$$

If $\delta \geq 2$, then Theorem 4 gives

$$\begin{aligned} \operatorname{diam} \Gamma &\leq \max\left\{2, \left\lfloor \frac{3|V(\Gamma)|-4}{\delta+1} \right\rfloor - 1\right\} \leq \max\left\{2, \left\lfloor \frac{3n-3(r-1)(\delta+1)-4}{\delta+1} \right\rfloor - 1\right\} \\ &= \max\left\{2, \left\lfloor \frac{3n-4}{\delta+1} \right\rfloor - 3r + 2\right\}. \end{aligned}$$

□

The arguments in the proof of Proposition 9 have the following consequence.

Corollary 3. *Let G be a graph with n vertices and minimum degree δ. Then G has at most $\lfloor \frac{n}{\delta+1} \rfloor$ connected components. In particular, if $\delta > \frac{n}{2} - 1$, then G is connected.*

Proposition 10. *Let G be a graph with n vertices, minimum degree δ and at least $r \geqslant 1$ connected components, and let Γ be a connected component of G.*

- *If* $\delta \geq (n-r)/(r+1)$, *then* $\operatorname{diam}\Gamma \leq 2$.
- *If* $\delta \geq (n-r-1)/(r+1)$, *then* $\operatorname{diam}\Gamma \leq 3$.
- *If* $\delta \geq (n-r-2)/(r+1)$, *then* $\operatorname{diam}\Gamma \leq 4$.
- *If* $\delta \geq (n-r-3)/(r+1)$, *then* $\operatorname{diam}\Gamma \leq 5$.
- *If* $\delta \geq (n-r-4)/(r+1)$, *then* $\operatorname{diam}\Gamma \leq 6$.
- *If* $\delta \geq (n-r-1)/(r+2)$, *then* $\operatorname{diam}\Gamma \leq 5$.
- *If* $\delta \geq (n-r-2)/(r+2)$, *then* $\operatorname{diam}\Gamma \leq 6$.

Proof. The arguments in the proof of Proposition 9 give that Γ has at most $n-(r-1)(\delta+1)$ vertices.

Assume first that $\delta \geq (n-r+1-j)/(r+1)$, with $1 \leqslant j \leqslant 5$. Thus,

$$\begin{aligned} \delta \geq \frac{n+2-j}{r+1} - 1 \quad &\Rightarrow \quad (r+1)(\delta+1) \geq n+2-j \quad \Rightarrow \\ 2\delta + (r-1)(\delta+1) \geq n-j \quad &\Rightarrow \quad \delta \geq \frac{n-(r-1)(\delta+1)-j}{2} \quad \Rightarrow \\ \delta \geq \frac{|V(\Gamma)|-j}{2}, \end{aligned}$$

and Theorem 5 gives $\operatorname{diam}\Gamma \leqslant j+1$, for each $1 \leqslant j \leqslant 5$.

Assume now that $\delta \geq (n-r-j+1)/(r+2)$, with $2 \leqslant j \leqslant 3$. Thus,

$$\begin{aligned} \delta \geq \frac{n-j+3}{r+2} - 1 \quad &\Rightarrow \quad (r+2)(\delta+1) \geq n-j+3 \quad \Rightarrow \\ 3\delta + (r-1)(\delta+1) \geq n-j \quad &\Rightarrow \quad \delta \geq \frac{n-(r-1)(\delta+1)-j}{3} \quad \Rightarrow \\ \delta \geq \frac{|V(\Gamma)|-j}{3}, \end{aligned}$$

and Theorem 5 gives $\operatorname{diam}\Gamma \leqslant j+3$, for $2 \leqslant j \leqslant 3$. □

The results that we are going to show in the following three propositions provide information about a graph based on its polynomial ID.

Proposition 11. *If G is a graph with at least $r \geqslant 1$ connected components, and Γ is a connected component of G.*

i. *If* $ID(G,0) \neq 0$ *then* $\operatorname{diam}\Gamma \leq ID(G,1) - 1 - (r-1)(\operatorname{Deg}_{\min} ID(G,x) + 2)$,

ii. *If* $ID(G,0) = 0$, *then*

$$\operatorname{diam}\Gamma \leq \max\left\{2, \left\lfloor \frac{3ID(G,1)-4}{\operatorname{Deg}_{\min} ID(G,x)+2} \right\rfloor - 3r + 2\right\}.$$

Proof. Using the results summarized in Proposition 1, we have that the number of vertices that have degree 1 is equal to $ID(G,0)$, while $ID(G,1)$ is equal to the number of total vertices of the graph. On the other hand, it is obtained directly that if $ID(G,0)$ is different from zero then the minimum degree δ of the graph G is exactly 1, while if $ID(G,0) = 0$ then the $\delta \geq 2$. Finally, from Proposition 8 we have $\operatorname{Deg} ID(G,x) = \Delta - 1$ and $\operatorname{Deg}_{\min} ID(G,x) = \delta - 1$, and directly applying the results of Proposition 9 we obtain the proposed result. □

Continuing with the same arguments used previously, from Proposition 1 we have that $ID(G,1) = n$, while from Proposition 8 we have that $\operatorname{Deg}_{\min} ID(G,x) + 1 = \delta$, then as direct consequence of Corollary 3 we obtain the following result.

Proposition 12. *Let G be a graph. Then G has at most*

$$\left\lfloor \frac{ID(G,1)}{\operatorname{Deg}_{\min} ID(G,x) + 2} \right\rfloor$$

connected components. In particular, if $2\operatorname{Deg}_{\min} ID(G,x) + 4 > ID(G,1)$*, then G is connected.*

In the same way, the next proposition is a direct consequence of the results given in Propositions 1, 8, and 10.

Proposition 13. *Let G be a graph with at least* $r \geqslant 1$ *connected components, and let* Γ *be a connected component of G.*

i. *If* $\operatorname{Deg}_{\min} ID(G,x) + 1 \geq \frac{(ID(G,1) - r + 1 - j)}{(r+1)}$, *then* $\operatorname{diam}\Gamma \leq j + 1$, $\forall j = 1, \ldots, 5$.

ii. *If* $\operatorname{Deg}_{\min} ID(G,x) + 1 \geq \frac{(ID(G,1) - r - 1)}{(r+2)}$, *then* $\operatorname{diam}\Gamma \leq 5$.

iii. *If* $\operatorname{Deg}_{\min} ID(G,x) + 1 \geq \frac{(ID(G,1) - r - 2)}{(r+2)}$, *then* $\operatorname{diam}\Gamma \leq 6$.

If $p(x)$ is a polynomial, then we denote by $\mathfrak{D}(p(x))$ the number of non-zero coefficients of $p(x)$. In particular, if $p(x) = ID(G,x)$, then $\mathfrak{D}(ID(G,x))$ is the cardinality of the set $\{d_u : u \in V(G)\}$.

Let us define inductively the graph Γ_k for $k \geqslant 2$. Let Γ_2 and Γ_3 be the path graphs P_2 and P_3, respectively. If $k \geqslant 4$, then we define Γ_k from Γ_{k-2} in the following way: Choose two points $u_1, u_{k-1} \notin V(\Gamma_{k-2})$ and define Γ_k by

$$V(\Gamma_k) = V(\Gamma_{k-2}) \cup \{u_1, u_{k-1}\}, \qquad E(\Gamma_k) = E(\Gamma_{k-2}) \cup \{u_{k-1}u_1\} \cup \{ \cup_{v \in V(\Gamma_{k-2})} u_{k-1}v \}.$$

Theorem 6. *If G is a graph with n vertices, minimum degree* δ *and at least r connected components, then the following statements hold:*

- $1 \leqslant \mathfrak{D}(ID(G,x)) \leqslant n - \delta - (r-1)(\delta+1)$,
- $\mathfrak{D}(ID(G,x)) = 1$ *if and only if G is regular,*
- $\mathfrak{D}(ID(G,x)) = n - 1$ *if and only if G is isomorphic to* Γ_n.

Proof. If G is a connected graph, then the inequalities $\delta \leqslant d_u \leqslant n - 1$ for every $u \in V(G)$ give $1 \leqslant \mathfrak{D}(ID(G,x)) \leqslant n - \delta$. Assume now that G is not connected and let Γ be a connected component of G. The argument in the proof of Proposition 9 gives that Γ has at most $n - (r-1)(\delta+1)$ vertices. Thus, $\delta \leqslant d_u \leqslant |V(\Gamma)| - 1$ for every $u \in V(\Gamma)$, and so, $\delta \leqslant d_u \leqslant n - 1 - (r-1)(\delta+1)$ for every $u \in V(\Gamma)$, and we conclude $1 \leqslant \mathfrak{D}(ID(G,x)) \leqslant n - \delta - (r-1)(\delta+1)$.

The second item is direct, since $\mathfrak{D}(ID(G,x)) = 1$ if and only if every vertex has the same degree.

Next, let us show by induction on n that the degree sequence $\{d_u : u \in V(\Gamma_n)\}$ of Γ_n is equal to $\{1, 2, \ldots, n-1\}$ for each $n \geqslant 2$ (if $d_{u_1} = d_{u_2}$ for some $u_1, u_2 \in V(\Gamma_n)$, then the value $d_{u_1} = d_{u_2}$ appears just once in the degree sequence). This is direct if $n = 2$ and $n = 3$. Assume that the statement holds for some Γ_{n-2} with $n \geqslant 4$, and let us prove it for Γ_n. If $v \in V(\Gamma_{n-2})$, then the definition of Γ_n gives $\deg_{\Gamma_n} v = \deg_{\Gamma_{n-2}} v + 1$. The induction hypothesis gives $\{\deg_{\Gamma_{n-2}} u : u \in V(\Gamma_{n-2})\} = \{1, 2, \ldots, n-3\}$, and so, $\{\deg_{\Gamma_n} u : u \in V(\Gamma_{n-2})\} = \{2, 3, \ldots, n-2\}$. Since $\deg_{\Gamma_n} u_1 = 1$ and $\deg_{\Gamma_n} u_{n-1} = n-1$, the degree sequence of Γ_n is $\{1, 2, \ldots, n-1\}$. Hence, $\mathfrak{D}(ID(\Gamma_n, x)) = n-1$ for every $n \geqslant 2$. Note that Γ_n is connected, since it contains a vertex with degree $n-1$ and $|V(\Gamma_n)| = n$.

Assume now that G is a graph with $\mathfrak{D}(ID(G, x)) = n-1$. If $n = 2$ and $n = 3$, it is clear that G is isomorphic to Γ_n. Assume that $n \geqslant 4$. Since $\{\deg_G u : u \in V(G)\} \subseteq \{1, 2, \ldots, n-1\}$ and $\mathfrak{D}(ID(G, x)) = n-1$, we have $\{\deg_G u : u \in V(G)\} = \{1, 2, \ldots, n-1\}$. Choose vertices $v_1, v_{n-1} \in V(G)$ with $\deg_G v_1 = 1$ and $\deg_G v_{n-1} = n-1$, and denote by G_{n-2} the graph with $n-2$ vertices obtained from G by deleting the vertices v_1, v_{n-1} and the edges incident to them. Note that v_1 is incident just to v_{n-1}. Since there exists $v \in V(G)$ with $\deg_G v = n-2$ (recall that $n-2 \geqslant 2 > 1$), we have $\deg_{G_{n-2}} v = n-3$, and so, $uv \in E(G_{n-2})$ for every $u \in V(G_{n-2}) \setminus \{v\}$. Thus, $\deg_{G_{n-2}} u \geqslant 1$ for every $u \in V(G_{n-2})$, and G_{n-2} is connected. Also, $\deg_{G_{n-2}} u = \deg_G u - 1$ for every $u \in V(G_{n-2})$. Hence, the degree sequence of G_{n-2} is $\{1, 2, \ldots, n-3\}$. If $2 \leqslant n-2 \leqslant 3$, then G_{n-2} is isomorphic to Γ_{n-2} and G is isomorphic to Γ_n. Otherwise we obtain, by iterating this process, a finite sequence of graphs $G_{n-2}, G_{n-4}, \ldots, G_{n-2k}$ with $2 \leqslant n-2k \leqslant 3$; thus, G_{n-2k} is isomorphic to Γ_{n-2k} and G is isomorphic to Γ_n. □

Theorem 6 has the following consequence.

Corollary 4. *If G is a graph with n vertices, then* $1 \leqslant \mathfrak{D}(ID(G, x)) \leqslant n-1$.

The argument in the proof of Theorem 6 also gives the following result.

Proposition 14. *If G is a graph with n vertices, maximum degree* Δ *and minimum degree* δ*, then* $\mathfrak{D}(ID(G, x)) \leqslant \Delta - \delta + 1$.

Next, we present a result that allows to bound the inverse degree index of a graph by using information about its ID polynomial.

Given a graph G, let us denote by k_Δ and k_δ the cardinality of the sets $\{u \in V(G) : d_u = \Delta\}$ and $\{u \in V(G) : d_u = \Delta\}$, respectively.

Proposition 15. *If G is a graph with n vertices, maximum degree* Δ *and minimum degree* δ*, then*

$$\frac{k_\delta}{\delta} + \frac{n-k_\delta}{\Delta} \leqslant ID(G) \leqslant \frac{k_\Delta}{\Delta} + \frac{n-k_\Delta}{\delta},$$

and the equality in each inequality is attained if and only if we have either that G is regular or $k_\delta + k_\Delta = n$.

Proof. Let us denote by k_j the cardinality of the set $\{u \in V(G) : d_u = j\}$. Thus,

$$ID(G) = \int_0^1 ID(G, x)\, dx = \int_0^1 \sum_{u \in V(G)} x^{d_u - 1}\, dx = \int_0^1 \sum_{j=\delta}^{\Delta} k_j\, x^{j-1}\, dx = \sum_{j=\delta}^{\Delta} \frac{k_j}{j}.$$

Hence,

$$ID(G) = \frac{k_\delta}{\delta} + \sum_{j=\delta+1}^{\Delta} \frac{k_j}{j} \geqslant \frac{k_\delta}{\delta} + \sum_{j=\delta+1}^{\Delta} \frac{k_j}{\Delta} = \frac{k_\delta}{\delta} + \frac{n-k_\delta}{\Delta},$$

$$ID(G) = \frac{k_\Delta}{\Delta} + \sum_{j=\delta}^{\Delta-1} \frac{k_j}{j} \leqslant \frac{k_\Delta}{\Delta} + \sum_{j=\delta}^{\Delta-1} \frac{k_j}{\delta} = \frac{k_\Delta}{\Delta} + \frac{n-k_\Delta}{\delta}.$$

If G is regular, then the lower and upper bounds are both equal to n/δ, and the equalities hold. If $k_\delta + k_\Delta = n$, then the lower and upper bounds are both equal to $k_\delta/\delta + k_\Delta/\Delta$, and the equalities hold.

Assume now that the equality is attained in the lower bound and G is not a regular graph. Thus, $\delta < \Delta$ and

$$\sum_{j=\delta+1}^{\Delta} \frac{k_j}{j} = \sum_{j=\delta+1}^{\Delta} \frac{k_j}{\Delta},$$

and so, $k_j = 0$ for every $\delta < j < \Delta$. Hence, $k_\delta + k_\Delta = n$.

If the equality is attained in the upper bound and G is not a regular graph, a similar argument also gives $k_\delta + k_\Delta = n$. □

Although two non-isomorphic graphs can have the same ID polynomial, we show now that two graphs with the same ID polynomial must be "similar".

For each function $f : \mathbb{N} \to (0,\infty)$, let us define the following topological indices

$$I_f(G) = \sum_{u\in V(G)} f(d_u), \qquad II_f(G) = \prod_{u\in V(G)} f(d_u).$$

Note that if $f(k) = k^\alpha$, then $I_f = M_1^\alpha$. The *Narumi–Katayama index* is defined in [20] as

$$NK(G) = \prod_{u\in V(G)} d_u.$$

If $f(t) = t$, then $II_f = NK$.

Theorem 7. *Let G and Γ be two graphs with $ID(G,x) = ID(\Gamma,x)$. Then $I_f(G) = I_f(\Gamma)$ and $II_f(G) = II_f(\Gamma)$ for every $f : \mathbb{N} \to (0,\infty)$. In particular, $M_1^\alpha(G) = M_1^\alpha(\Gamma)$ for every $\alpha \in \mathbb{R}$, and $NK(G) = NK(\Gamma)$.*

Proof. As in the proof of Proposition 15, for any graph G_0 with maximum degree Δ and minimum degree δ, and $\delta \leqslant j \leqslant \Delta$, let us define $k_j(G_0)$ as the cardinality of $\{u \in V(G_0) \mid d_u = j\}$. We have seen that

$$ID(G,x) = \sum_{j=\delta}^{\Delta} k_j(G_0)\, x^{j-1}.$$

Thus, if $ID(G,x) = ID(\Gamma,x)$, we conclude that $k_j(G) = k_j(\Gamma)$ for every $\delta \leqslant j \leqslant \Delta$. Note that we have, for any graph G_0 and any function $f : \mathbb{N} \to (0,\infty)$,

$$I_f(G_0) = \sum_{j=\delta}^{\Delta} k_j(G_0) f(j), \qquad II_f(G_0) = \prod_{j=\delta}^{\Delta} f(j)^{k_j(G_0)}.$$

Hence, we conclude that $I_f(G) = I_f(\Gamma)$ and $II_f(G) = II_f(\Gamma)$. □

We also have a relation between the harmonic and the inverse degree polynomials.

Given a graph G, the *line graph* of G is denoted by $\mathcal{L}(G)$. It is not difficult to verify that if $uv \in E(G)$, then the degree of $w_{uv} \in V(\mathcal{L}(G))$ is $d_u + d_v - 2$.

Line graphs were initially introduced by Whitney in [21] and a few year later by Krausz in his work [22], however the first time that the exact terminology of line graph was used, it was in the joint work by Harary and Norman in [23]. Nowadays this is an active topic, where we can find very recent works that study some topological indices on line graphs (see, e.g., [24,25]).

Proposition 16. *Let G be a graph. Then,*

$$H(G,x) = x^2 ID(\mathcal{L}(G),x), \qquad ID(\mathcal{L}(G)) = \int_0^1 \frac{H(G,x)}{x^2}\, dx.$$

Proof. We have

$$H(G,x) = \sum_{uv\in E(G)} x^{d_u+d_v-1} = \sum_{w\in V(\mathcal{L}(G))} x^{d_w+1} = x^2 \sum_{w\in V(\mathcal{L}(G))} x^{d_w-1} = x^2 ID(\mathcal{L}(G),x),$$

$$ID(\mathcal{L}(G)) = \int_0^1 ID(\mathcal{L}(G),x)\,dx = \int_0^1 \frac{H(G,x)}{x^2}\,dx.$$

If G is a regular graph is regular, then the both bounds are the same, and they are equal to $GA_1(G)$. If we have the equality, then $4(d_u+d_v)^{-2} = \Delta^{-2}$ for every $uv \in E(G)$; thus, G is a regular graph. □

4. Conclusions

In this work we obtain several mathematical relations of the inverse degree polynomial from the symmetry property of the inverse degree index. We show that some properties of graphs, such as the cardinality of the set of vertices and edges, or the cyclomatic number, can be deduced from their inverse degree polynomials. We also obtain bounds of the ID index in terms of the values of the ID polynomial and we have a relation between the harmonic and the inverse degree polynomials.

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