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On new Milne-type inequalities and applications

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Abstract

Inequalities play a major role in pure and applied mathematics. In particular, the inequality plays an important role in the study of Rosseland's integral for the stellar absorption. In this paper we obtain new Milne-type inequalities, and we apply them to the generalized Riemann–Liouville-type integral operators, which include most of the known Riemann–Liouville integral operators.

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1 Introduction

Integral inequalities are used in countless mathematical problems such as approximation theory and spectral analysis, statistical analysis, and the theory of distributions. Studies involving integral inequalities play an important role in several areas of science and engineering.

In recent years there has been a growing interest in the study of many classical inequalities applied to integral operators associated with different types of fractional derivatives, since integral inequalities and their applications play a vital role in the theory of differential equations and applied mathematics. Some of the inequalities studied are Gronwall, Chebyshev, Hermite–Hadamard-type, Ostrowski-type, Grüss-type, Hardy-type, Gagliardo–Nirenberg-type, Jensen-type, Opial-type, Milne-type, reverse Minkowski, and reverse Hölder inequalities (see, e.g., [1, 3, 4, 7–9, 11, 12, 14–20]).

In this work we obtain new Milne-type inequalities, and we apply them to the generalized Riemann–Liouville-type integral operators defined in [2], which include most of the known Riemann–Liouville integral operators.

2 Preliminaries

One of the first operators that can be called fractional is the Riemann–Liouville fractional derivative of order $\alpha \in \mathbb{C}$, with $\mathrm{Re}(\alpha) > 0$, defined as follows (see [6]).

Definition 1 Let $a < b$ and $f \in L^1((a,b);\mathbb{R})$. The *right- and left-side Riemann–Liouville fractional integrals of order* α, with $\operatorname{Re}(\alpha) > 0$, are defined, respectively, by

$$ {}^{RL}J_{a^+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s)\, ds, \tag{1} $$

and

$$ {}^{RL}J_{b^-}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (s-t)^{\alpha-1} f(s)\, ds, \tag{2} $$

with $t \in (a,b)$.

When $\alpha \in (0,1)$, their corresponding *Riemann–Liouville fractional derivatives* are given by

$$ \begin{aligned} \left({}^{RL}D_{a^+}^{\alpha} f\right)(t) &= \frac{d}{dt}\left({}^{RL}J_{a^+}^{1-\alpha} f(t)\right) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t \frac{f(s)}{(t-s)^{\alpha}}\, ds, \\ \left({}^{RL}D_{b^-}^{\alpha} f\right)(t) &= -\frac{d}{dt}\left({}^{RL}J_{b^-}^{1-\alpha} f(t)\right) = -\frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_t^b \frac{f(s)}{(s-t)^{\alpha}}\, ds. \end{aligned} $$

Other definitions of fractional operators are the following ones.

Definition 2 Let $a < b$ and $f \in L^1((a,b);\mathbb{R})$. The *right- and left-side Hadamard fractional integrals of order* α, with $\operatorname{Re}(\alpha) > 0$, are defined, respectively, by

$$ H_{a^+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{f(s)}{s}\, ds, \tag{3} $$

and

$$ H_{b^-}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b \left(\log \frac{s}{t}\right)^{\alpha-1} \frac{f(s)}{s}\, ds, \tag{4} $$

with $t \in (a,b)$.

When $\alpha \in (0,1)$, the *Hadamard fractional derivatives* are given by the following expressions:

$$ \begin{aligned} \left({}^{H}D_{a^+}^{\alpha} f\right)(t) &= t\frac{d}{dt}\left(H_{a^+}^{1-\alpha} f(t)\right) = \frac{1}{\Gamma(1-\alpha)} t \frac{d}{dt} \int_a^t \left(\log \frac{t}{s}\right)^{-\alpha} \frac{f(s)}{s}\, ds, \\ \left({}^{H}D_{b^-}^{\alpha} f\right)(t) &= -t\frac{d}{dt}\left(H_{b^-}^{1-\alpha} f(t)\right) = \frac{-1}{\Gamma(1-\alpha)} t \frac{d}{dt} \int_t^b \left(\log \frac{s}{t}\right)^{-\alpha} \frac{f(s)}{s}\, ds, \end{aligned} $$

with $t \in (a,b)$.

Definition 3 Let $0 < a < b$, $g : [a,b] \to \mathbb{R}$ be an increasing positive function on $(a,b]$ with continuous derivative on (a,b), $f : [a,b] \to \mathbb{R}$ an integrable function, and $\alpha \in (0,1)$ a fixed real number. The right- and left-side fractional integrals in [10] of order α of f with respect to g are defined, respectively, by

$$I^{\alpha}_{g,a^+}f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{g'(s)f(s)}{(g(t)-g(s))^{1-\alpha}}\, ds, \tag{5}$$

and

$$I^{\alpha}_{g,b^-}f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b \frac{g'(s)f(s)}{(g(s)-g(t))^{1-\alpha}}\, ds, \tag{6}$$

with $t \in (a,b)$.

There are other definitions of integral operators in the global case, but they are slight modifications of the previous ones.

3 General fractional integral of Riemann–Liouville type

Now, we give the definition of a general fractional integral in [2].

Definition 4 Let $a < b$ and $\alpha \in \mathbb{R}^+$. Let $g : [a,b] \to \mathbb{R}$ be a positive function on $(a,b]$ with continuous positive derivative on (a,b), and $G : [0, g(b) - g(a)] \times (0,\infty) \to \mathbb{R}$ a continuous function that is positive on $(0, g(b) - g(a)] \times (0,\infty)$. Let us define the function $T : [a,b] \times [a,b] \times (0,\infty) \to \mathbb{R}$ by

$$T(t,s,\alpha) = \frac{G(|g(t)-g(s)|,\alpha)}{g'(s)}.$$

The *right and left integral operators*, denoted, respectively, by J^{α}_{T,a^+} and J^{α}_{T,b^-}, are defined for each measurable function f on $[a,b]$ as

$$J^{\alpha}_{T,a^+}f(t) = \int_a^t \frac{f(s)}{T(t,s,\alpha)}\, ds, \tag{7}$$

$$J^{\alpha}_{T,b^-}f(t) = \int_t^b \frac{f(s)}{T(t,s,\alpha)}\, ds, \tag{8}$$

with $t \in [a,b]$.

We say that $f \in L^1_T[a,b]$ if $J^{\alpha}_{T,a^+}|f|(t), J^{\alpha}_{T,b^-}|f|(t) < \infty$ for every $t \in [a,b]$.

Note that these operators generalize the integral operators in Definitions 1, 2, and 3:
(A) If we choose

$$g(t) = t, \qquad G(x,\alpha) = \Gamma(\alpha)x^{1-\alpha}, \qquad T(t,s,\alpha) = \Gamma(\alpha)|t-s|^{1-\alpha},$$

then J^{α}_{T,a^+} and J^{α}_{T,b^-} are the right and left Riemann–Liouville fractional integrals ${}^{RL}J^{\alpha}_{a^+}$ and ${}^{RL}J^{\alpha}_{b^-}$ in (1) and (2), respectively. The corresponding right and left Riemann–Liouville fractional derivatives are

$$\left({}^{RL}D^{\alpha}_{a^+}f\right)(t) = \frac{d}{dt}\left({}^{RL}J^{1-\alpha}_{a^+}f(t)\right), \qquad \left({}^{RL}D^{\alpha}_{b^-}f\right)(t) = -\frac{d}{dt}\left({}^{RL}J^{1-\alpha}_{b^-}f(t)\right).$$

(B) If we choose

$$g(t)=\log t,\qquad G(x,\alpha)=\Gamma(\alpha)x^{1-\alpha},\qquad T(t,s,\alpha)=\Gamma(\alpha)t\left|\log\frac{t}{s}\right|^{1-\alpha},$$

then J^{α}_{T,a^+} and J^{α}_{T,b^-} are the right and left Hadamard fractional integrals $H^{\alpha}_{a^+}$ and $H^{\alpha}_{b^-}$ in (3) and (4), respectively. The corresponding right and left Hadamard fractional derivatives are

$$\left({}^{H}D^{\alpha}_{a^+}f\right)(t)=t\frac{d}{dt}\left(H^{1-\alpha}_{a^+}f(t)\right),\qquad \left({}^{H}D^{\alpha}_{b^-}f\right)(t)=-t\frac{d}{dt}\left(H^{1-\alpha}_{b^-}f(t)\right).$$

(C) If we choose a function g with the properties in Definition 4 and

$$G(x,\alpha)=\Gamma(\alpha)x^{1-\alpha},\qquad T(t,s,\alpha)=\Gamma(\alpha)\frac{|g(t)-g(s)|^{1-\alpha}}{g'(s)},$$

then J^{α}_{T,a^+} and J^{α}_{T,b^-} are the right and left fractional integrals I^{α}_{g,a^+} and I^{α}_{g,b^-} in (5) and (6), respectively.

Definition 5 Let $a<b$ and $\alpha\in\mathbb{R}^+$. Let $g:[a,b]\to\mathbb{R}$ be a positive function on $(a,b]$ with continuous positive derivative on (a,b), and $G:[0,g(b)-g(a)]\times(0,\infty)\to\mathbb{R}$ a continuous function that is positive on $(0,g(b)-g(a)]\times(0,\infty)$. For each function $f\in L^1_T[a,b]$, its *right and left generalized derivative of order* α are defined, respectively, by

$$\begin{aligned}D^{\alpha}_{T,a^+}f(t)&=\frac{1}{g'(t)}\frac{d}{dt}\left(J^{1-\alpha}_{T,a^+}f(t)\right),\\ D^{\alpha}_{T,b^-}f(t)&=\frac{-1}{g'(t)}\frac{d}{dt}\left(J^{1-\alpha}_{T,b^-}f(t)\right),\end{aligned}\tag{9}$$

for each $t\in(a,b)$.

Note that if we choose

$$g(t)=t,\qquad G(x,\alpha)=\Gamma(\alpha)x^{1-\alpha},\qquad T(t,s,\alpha)=\Gamma(\alpha)|t-s|^{1-\alpha},$$

then $D^{\alpha}_{T,a^+}f(t)={}^{RL}D^{\alpha}_{a^+}f(t)$ and $D^{\alpha}_{T,b^-}f(t)={}^{RL}D^{\alpha}_{b^-}f(t)$. Also, we can obtain Hadamard and others fractional derivatives as particular cases of this generalized derivative.

4 Milne-type inequalities

Milne proved in 1925 the two following discrete and continuous versions of a useful inequality [13]:

Proposition 6 *The following inequality holds for every* $a_i,b_i>0$ *for* $1\le i\le n$:

$$\sum_{i=1}^{n}a_i\cdot\sum_{i=1}^{n}b_i\ge\sum_{i=1}^{n}(a_i+b_i)\cdot\sum_{i=1}^{n}\frac{a_ib_i}{a_i+b_i}.$$

Remark 7 Since

$$\frac{a_i b_i}{a_i + b_i} \leq \frac{1}{4}(a_i + b_i),$$

the conclusion of Proposition 6 also holds for every $a_i, b_i \geq 0$ with the convention $0 \cdot 0/(0 + 0) = 0$.

Proposition 8 *Let $\phi : (0,\infty) \to [0,\infty)$ be a Riemann integrable function with $\int_0^\infty \phi(x)\,dx = 1$. Let $a_i > 0$ and $f_i : (0,\infty) \to (0,\infty)$ such that ϕ/f_i is a Riemann integrable function on $(0,\infty)$ for $1 \leq i \leq n$. Then,*

$$\frac{1}{\int_0^\infty \frac{\phi(x)\,dx}{a_1 f_1(x)+\cdots+a_n f_n(x)}} \geq \frac{a_1}{\int_0^\infty \frac{\phi(x)\,dx}{f_1(x)}} + \cdots + \frac{a_n}{\int_0^\infty \frac{\phi(x)\,dx}{f_n(x)}}.$$

We start with our general version of Proposition 6.

Theorem 9 *Let $x_i, y_j \geq 0$, $c_{i,j} > 0$ for any $i, j \geq 1$. If $\sum_{i=1}^\infty x_i > 0$ and $\sum_{j=1}^\infty y_j > 0$, then*

$$\frac{1}{\sum_{i=1}^\infty \frac{x_i}{\sum_{j=1}^\infty c_{i,j} y_j}} \geq \sum_{j=1}^\infty \frac{y_j}{\sum_{i=1}^\infty \frac{x_i}{c_{i,j}}}.$$

Proof Let us prove first

$$\frac{1}{\sum_{i=1}^m \frac{x_i}{\sum_{j=1}^n c_{i,j} y_j}} \geq \sum_{j=1}^n \frac{y_j}{\sum_{i=1}^m \frac{x_i}{c_{i,j}}}, \tag{10}$$

if $x_i, y_j > 0$ for $1 \leq i \leq m$, $1 \leq j \leq n$.

If we define $k_{i,j} := c_{i,j} y_j / x_i$, then it suffices to prove that

$$\frac{1}{\sum_{i=1}^m \frac{1}{\sum_{j=1}^n k_{i,j}}} \geq \sum_{j=1}^n \frac{1}{\sum_{i=1}^m \frac{1}{k_{i,j}}}. \tag{11}$$

Let us prove (11) by induction on n.

If $n = 1$, then the inequality (11) holds since, in fact, it is an equality.

If $n = 2$, $a_i := 1/k_{i,1}$ and $b_i := 1/k_{i,2}$, then the following inequalities are equivalent

$$\frac{1}{\sum_{i=1}^m \frac{1}{k_{i,1}+k_{i,2}}} \geq \frac{1}{\sum_{i=1}^m \frac{1}{k_{i,1}}} + \frac{1}{\sum_{i=1}^m \frac{1}{k_{i,2}}},$$

$$\frac{1}{\sum_{i=1}^m \frac{a_i b_i}{a_i+b_i}} = \frac{1}{\sum_{i=1}^m \frac{1}{1/a_i+1/b_i}} \geq \frac{1}{\sum_{i=1}^m a_i} + \frac{1}{\sum_{i=1}^m b_i} = \frac{\sum_{i=1}^m (a_i + b_i)}{\sum_{i=1}^m a_i \cdot \sum_{i=1}^m b_i},$$

$$\sum_{i=1}^m a_i \cdot \sum_{i=1}^m b_i \geq \sum_{i=1}^m (a_i + b_i) \cdot \sum_{i=1}^m \frac{a_i b_i}{a_i + b_i},$$

and this last inequality holds by Proposition 6.

Finally, assume that (11) holds for $n-1 \geq 2$. Then, the induction hypothesis and the previous inequality give

$$\begin{aligned}
\frac{1}{\sum_{i=1}^{m} \frac{1}{\sum_{j=1}^{n} k_{i,j}}} &= \frac{1}{\sum_{i=1}^{m} \frac{1}{\sum_{j=1}^{n-2} k_{i,j}+(k_{i,n-1}+k_{i,n})}} \\
&\geq \sum_{j=1}^{n-2} \frac{1}{\sum_{i=1}^{m} \frac{1}{k_{i,j}}} + \frac{1}{\sum_{i=1}^{m} \frac{1}{k_{i,n-1}+k_{i,n}}} \\
&\geq \sum_{j=1}^{n-2} \frac{1}{\sum_{i=1}^{m} \frac{1}{k_{i,j}}} + \frac{1}{\sum_{i=1}^{m} \frac{1}{k_{i,n-1}}} + \frac{1}{\sum_{i=1}^{m} \frac{1}{k_{i,n}}} \\
&= \sum_{j=1}^{n} \frac{1}{\sum_{i=1}^{m} \frac{1}{k_{i,j}}},
\end{aligned}$$

which completes the proof of (11). Hence, (10) holds if $x_i, y_j > 0$ for $1 \leq i \leq m$, $1 \leq j \leq n$.

If we take limits as $x_i \to 0$ and/or $y_j \to 0$ for some indices $1 \leq i \leq m$, $1 \leq j \leq n$ in (10), we obtain the same conclusion if $x_i, y_j \geq 0$ for $1 \leq i \leq m$, $1 \leq j \leq n$, $\sum_{i=1}^{m} x_i > 0$ and $\sum_{j=1}^{n} y_j > 0$.

If we take limits as $m \to \infty$ in (10), then we obtain for every n

$$\frac{1}{\sum_{i=1}^{\infty} \frac{x_i}{\sum_{j=1}^{n} c_{i,j} y_j}} \geq \sum_{j=1}^{n} \frac{y_j}{\sum_{i=1}^{\infty} \frac{x_i}{c_{i,j}}},$$

if $x_i, y_j \geq 0$ for $i \geq 1$, $1 \leq j \leq n$, $\sum_{i=1}^{\infty} x_i > 0$ and $\sum_{j=1}^{n} y_j > 0$. Since

$$\frac{1}{\sum_{i=1}^{\infty} \frac{x_i}{\sum_{j=1}^{\infty} c_{i,j} y_j}} \geq \frac{1}{\sum_{i=1}^{\infty} \frac{x_i}{\sum_{j=1}^{n} c_{i,j} y_j}}$$

for every n, we have

$$\frac{1}{\sum_{i=1}^{\infty} \frac{x_i}{\sum_{j=1}^{\infty} c_{i,j} y_j}} \geq \sum_{j=1}^{n} \frac{y_j}{\sum_{i=1}^{\infty} \frac{x_i}{c_{i,j}}}$$

for every n. Then, the result follows if we take limits as $n \to \infty$ in this last inequality. □

Remark 10 The argument in the proof of Theorem 9 gives that Proposition 6 is equivalent to the case $n = 2$ in (10). Hence, this theorem is a generalization of the discrete Milne inequality.

Corollary 11 *Let $x_i, y_j \geq 0$, $c_{i,j} > 0$ for $1 \leq i \leq m$, $1 \leq j \leq n$. If $\sum_{i=1}^{m} x_i > 0$ and $\sum_{j=1}^{n} y_j > 0$, then*

$$\frac{1}{\sum_{i=1}^{m} \frac{x_i}{\sum_{j=1}^{n} c_{i,j} y_j}} \geq \sum_{j=1}^{n} \frac{y_j}{\sum_{i=1}^{m} \frac{x_i}{c_{i,j}}}.$$

In order to prove Theorem 20 below, generalizing Proposition 8 (the continuous version of Milne inequality), we need the following technical results.

Proposition 12 *Let $M > 0$, μ, ν two σ-finite measures on the spaces X, Y, respectively, and $f_n, f : X \times Y \to [M, \infty)$ measurable functions. If $\lim_{n\to\infty} f_n = f$ $(\mu \times \nu)$-a.e. and $f_n \le g \in L^1(\mu \times \nu)$, then*

$$\lim_{n\to\infty} \int_X \frac{d\mu(x)}{\int_Y f_n(x,y)\,d\nu(y)} = \int_X \frac{d\mu(x)}{\int_Y f(x,y)\,d\nu(y)}. \tag{12}$$

Proof Since

$$\begin{aligned}
&\left|\int_X \frac{d\mu(x)}{\int_Y f_n(x,y)\,d\nu(y)} - \int_X \frac{d\mu(x)}{\int_Y f(x,y)\,d\nu(y)}\right| \\
&\quad = \left|\int_X \frac{\int_Y (f(x,y) - f_n(x,y))\,d\nu(y)}{\int_Y f(x,y)\,d\nu(y) \int_Y f_n(x,y)\,d\nu(y)}\,d\mu(x)\right| \\
&\quad \le \int_X \frac{\int_Y |f(x,y) - f_n(x,y)|\,d\nu(y)}{\int_Y f(x,y)\,d\nu(y) \int_Y f_n(x,y)\,d\nu(y)}\,d\mu(x) \\
&\quad \le \int_X \frac{\int_Y |f(x,y) - f_n(x,y)|\,d\nu(y)}{\int_Y M\,d\nu(y) \int_Y M\,d\nu(y)}\,d\mu(x) \\
&\quad = \frac{1}{M^2 \nu(Y)^2} \int_X \int_Y \bigl|f(x,y) - f_n(x,y)\bigr|\,d\nu(y)\,d\mu(x),
\end{aligned}$$

$|f - f_n| \le f + f_n \le 2g \in L^1(\mu \times \nu)$ and $\lim_{n\to\infty} |f - f_n| = 0$ $(\mu \times \nu)$-a.e., the dominated convergence theorem gives

$$\begin{aligned}
0 &\le \liminf_{n\to\infty} \left|\int_X \frac{d\mu(x)}{\int_Y f_n(x,y)\,d\nu(y)} - \int_X \frac{d\mu(x)}{\int_Y f(x,y)\,d\nu(y)}\right| \\
&\le \limsup_{n\to\infty} \left|\int_X \frac{d\mu(x)}{\int_Y f_n(x,y)\,d\nu(y)} - \int_X \frac{d\mu(x)}{\int_Y f(x,y)\,d\nu(y)}\right| \\
&\le \lim_{n\to\infty} \frac{1}{M^2 \nu(Y)^2} \int_X \int_Y \bigl|f(x,y) - f_n(x,y)\bigr|\,d\nu(y)\,d\mu(x) = 0,
\end{aligned}$$

and this completes the proof. □

Proposition 13 *Let $0 < M \le N < \infty$, μ, ν be two finite measures on the spaces X, Y, respectively, and $f_n, f : X \times Y \to [M, N]$ measurable functions. If $\lim_{n\to\infty} f_n = f$ $(\mu \times \nu)$-a.e., then*

$$\lim_{n\to\infty} \int_Y \frac{d\nu(y)}{\int_X \frac{d\mu(x)}{f_n(x,y)}} = \int_Y \frac{d\nu(y)}{\int_X \frac{d\mu(x)}{f(x,y)}}. \tag{13}$$

Proof Since

$$\begin{aligned}
&\left|\int_Y \frac{d\nu(y)}{\int_X \frac{d\mu(x)}{f_n(x,y)}} - \int_Y \frac{d\nu(y)}{\int_X \frac{d\mu(x)}{f(x,y)}}\right| \\
&\quad = \left|\int_Y \frac{\int_X \frac{d\mu(x)}{f(x,y)} - \int_X \frac{d\mu(x)}{f_n(x,y)}}{\int_X \frac{d\mu(x)}{f_n(x,y)} \int_X \frac{d\mu(x)}{f(x,y)}}\,d\nu(y)\right| \\
&\quad \le \int_Y \frac{|\int_X \frac{d\mu(x)}{f(x,y)} - \int_X \frac{d\mu(x)}{f_n(x,y)}|}{\int_X \frac{d\mu(x)}{N} \int_X \frac{d\mu(x)}{N}}\,d\nu(y)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{N^2}{\mu(X)^2} \int_Y \left| \int_X \frac{d\mu(x)}{f(x,y)} - \int_X \frac{d\mu(x)}{f_n(x,y)} \right| d\nu(y) \\
&\leq \frac{N^2}{\mu(X)^2} \int_Y \int_X \frac{|f(x,y) - f_n(x,y)|}{f(x,y) f_n(x,y)} \, d\mu(x)\, d\nu(y) \\
&\leq \frac{N^2}{M^2 \mu(X)^2} \int_Y \int_X \left| f(x,y) - f_n(x,y) \right| d\mu(x)\, d\nu(y),
\end{aligned}
$$

$|f - f_n| \leq 2N \in L^1(\mu \times \nu)$ and $\lim_{n\to\infty} |f - f_n| = 0$ $(\mu \times \nu)$-a.e., the dominated convergence theorem gives the result. □

Proposition 14 *Let μ, ν be two measures on the spaces X, Y, respectively, and $f_n : X \times Y \to [0,\infty]$ measurable functions with $f_n \leq f_{n+1}$ for every n, and let $f := \lim_{n\to\infty} f_n$. If*

$$
\frac{1}{\int_Y f_1(x,y)\, d\nu(y)} \in L^1(\mu) \quad \Leftrightarrow \quad \frac{1}{\int_Y f(x,y)\, d\nu(y)} \in L^1(\mu),
$$

then

$$
\lim_{n\to\infty} \int_X \frac{d\mu(x)}{\int_Y f_n(x,y)\, d\nu(y)} = \int_X \frac{d\mu(x)}{\int_Y f(x,y)\, d\nu(y)}.
$$

Proof The monotone convergence theorem gives

$$
\lim_{n\to\infty} \int_Y f_n(x,y)\, d\nu(y) = \int_Y f(x,y)\, d\nu(y)
$$

for every $x \in X$.

Since $f_n \leq f$ for every n, if

$$
\frac{1}{\int_Y f(x,y)\, d\nu(y)} \notin L^1(\mu) \quad \Rightarrow \quad \frac{1}{\int_Y f_n(x,y)\, d\nu(y)} \notin L^1(\mu)
$$

for every n then,

$$
\lim_{n\to\infty} \int_X \frac{d\mu(x)}{\int_Y f_n(x,y)\, d\nu(y)} = \infty = \int_X \frac{d\mu(x)}{\int_Y f(x,y)\, d\nu(y)}.
$$

If

$$
\frac{1}{\int_Y f(x,y)\, d\nu(y)} \in L^1(\mu) \quad \Rightarrow \quad \frac{1}{\int_Y f_1(x,y)\, d\nu(y)} \in L^1(\mu),
$$

by hypothesis. Since $f_1 \leq f_n$ for every n, we have

$$
0 \leq \frac{1}{\int_Y f_n(x,y)\, d\nu(y)} \leq \frac{1}{\int_Y f_1(x,y)\, d\nu(y)} \in L^1(\mu),
$$

and the dominated convergence theorem gives the result. □

Proposition 15 *Let μ, ν be two measures on the spaces X, Y, respectively, and $f_n : X \times Y \to [0,\infty]$ measurable functions with $f_n \geq f_{n+1}$ for every n, and let $f := \lim_{n\to\infty} f_n$. If*

$$
f_1(x,y) \in L^1(\nu) \quad \Leftrightarrow \quad f(x,y) \in L^1(\nu)
$$

for each $x \in X$, then

$$\lim_{n\to\infty} \int_X \frac{d\mu(x)}{\int_Y f_n(x,y)\,d\nu(y)} = \int_X \frac{d\mu(x)}{\int_Y f(x,y)\,d\nu(y)}.$$

Proof Fix $x \in X$. If $f(x,y) \notin L^1(\nu)$, then $f_n(x,y) \notin L^1(\nu)$ for every n and so,

$$\lim_{n\to\infty} \int_Y f_n(x,y)\,d\nu(y) = \infty = \int_Y f(x,y)\,d\nu(y).$$

If $f(x,y) \in L^1(\nu)$, then $f_1(x,y) \in L^1(\nu)$ and the dominated convergence theorem gives

$$\lim_{n\to\infty} \int_Y f_n(x,y)\,d\nu(y) = \int_Y f(x,y)\,d\nu(y).$$

Since

$$0 \le \frac{1}{\int_Y f_n(x,y)\,d\nu(y)} \le \frac{1}{\int_Y f_{n+1}(x,y)\,d\nu(y)}$$

for every n, the monotone convergence theorem gives the result. □

Proposition 16 *Let μ, ν be two measures on the spaces X, Y, respectively, and $f_n : X \times Y \to [0,\infty]$ measurable functions with $f_n \le f_{n+1}$ for every n, and let $f := \lim_{n\to\infty} f_n$. If*

$$\frac{1}{f_1(x,y)} \in L^1(\mu) \quad \Leftrightarrow \quad \frac{1}{f(x,y)} \in L^1(\mu)$$

for each $y \in Y$, then

$$\lim_{n\to\infty} \int_Y \frac{d\nu(y)}{\int_X \frac{d\mu(x)}{f_n(x,y)}} = \int_Y \frac{d\nu(y)}{\int_X \frac{d\mu(x)}{f(x,y)}}.$$

Proof Fix $y \in Y$. If $1/f(x,y) \notin L^1(\mu)$, then $1/f_n(x,y) \notin L^1(\mu)$ for every n and hence,

$$\lim_{n\to\infty} \int_X \frac{d\mu(x)}{f_n(x,y)} = \infty = \int_X \frac{d\mu(x)}{f(x,y)}.$$

If $1/f(x,y) \in L^1(\mu)$, then $1/f_1(x,y) \in L^1(\mu)$ and

$$\frac{1}{f_n(x,y)} \le \frac{1}{f_1(x,y)} \in L^1(\mu),$$

and the dominated convergence theorem gives

$$\lim_{n\to\infty} \int_X \frac{d\mu(x)}{f_n(x,y)} = \int_X \frac{d\mu(x)}{f(x,y)}.$$

Since

$$0 \le \frac{1}{\int_X \frac{d\mu(x)}{f_n(x,y)}} \le \frac{1}{\int_X \frac{d\mu(x)}{f_{n+1}(x,y)}}$$

for every n, the monotone convergence theorem gives the result. □

Proposition 17 *Let μ, ν be two measures on the spaces X, Y, respectively, and $f_n : X \times Y \to [0,\infty]$ measurable functions with $f_n \geq f_{n+1}$ for every n, and let $f := \lim_{n\to\infty} f_n$. Then,*

$$\lim_{n\to\infty} \int_Y \frac{d\nu(y)}{\int_X \frac{d\mu(x)}{f_n(x,y)}} \geq \int_Y \frac{d\nu(y)}{\int_X \frac{d\mu(x)}{f(x,y)}}.$$

Proof Since

$$0 \leq \int_X \frac{d\mu(x)}{f_n(x,y)} \leq \int_X \frac{d\mu(x)}{f_{n+1}(x,y)}$$

for every n and $y \in Y$, the monotone convergence theorem gives

$$\lim_{n\to\infty} \int_X \frac{d\mu(x)}{f_n(x,y)} = \int_X \frac{d\mu(x)}{f(x,y)}$$

for every $y \in Y$.

Since

$$\frac{1}{\int_X \frac{d\mu(x)}{f_n(x,y)}} \geq \frac{1}{\int_X \frac{d\mu(x)}{f_{n+1}(x,y)}}$$

for every n and $y \in Y$, there exists the limit

$$\lim_{n\to\infty} \int_Y \frac{d\nu(y)}{\int_X \frac{d\mu(x)}{f_n(x,y)}}$$

for every $y \in Y$.

Since

$$\frac{1}{\int_X \frac{d\mu(x)}{f_n(x,y)}} \geq 0$$

for every n and $y \in Y$, the Fatou lemma gives

$$\begin{aligned} \lim_{n\to\infty} \int_Y \frac{d\nu(y)}{\int_X \frac{d\mu(x)}{f_n(x,y)}} &= \liminf_{n\to\infty} \int_Y \frac{d\nu(y)}{\int_X \frac{d\mu(x)}{f_n(x,y)}} \\ &\geq \int_Y \liminf_{n\to\infty} \frac{1}{\int_X \frac{d\mu(x)}{f_n(x,y)}}\, d\nu(y) \\ &= \int_Y \frac{d\nu(y)}{\int_X \frac{d\mu(x)}{f(x,y)}}. \end{aligned}$$

□

Let us recall some background in [5]. A measure μ defined on the σ-algebra of all Borel sets in a locally compact Hausdorff space X is called a *Borel measure* on X. The measure μ is called *outer regular* on E if

$$\mu(E) = \inf\{\mu(U) : E \subseteq U, U \text{ open}\}$$

and *inner regular* on E if

$$\mu(E) = \sup\{\mu(K) : K \subseteq E, K \text{ compact}\}.$$

A *Radon measure* on X is a Borel measure that is finite on all compact sets, outer regular on all Borel sets, and inner regular on all open sets. Radon measures are also inner regular on all of their σ-finite sets [5, p. 216].

Recall that a second-countable space is a topological space whose topology has a countable base. In a second-countable, locally compact Hausdorff space, any Borel measure that is finite on compact sets is regular and hence Radon [5, p. 217].

The following two results appear in [5, pp. 217, 226].

Proposition 18 *If μ is a Radon measure on X, then $C_c(X)$ is dense in $L^p(\mu)$ for $1 \le p < \infty$.*

Proposition 19 *If X, Y are second-countable, locally compact Hausdorff spaces and μ, ν are Radon measures on X and Y, respectively, then $\mu \times \nu$ is a Radon measure on $X \times Y$.*

We can prove now the main result in this work, generalizing Proposition 8, the continuous version of the Milne inequality.

Theorem 20 *Let X, Y be second-countable, locally compact metric spaces, and let μ, ν be Borel measures on the metric spaces X, Y, respectively, which are finite on compact sets. If $f : X \times Y \to [0,\infty]$ is any measurable function, then*

$$\frac{1}{\int_X \frac{d\mu(x)}{\int_Y f(x,y)\,d\nu(y)}} \ge \int_Y \frac{d\nu(y)}{\int_X \frac{d\mu(x)}{f(x,y)}}. \tag{14}$$

Proof Since X, Y are σ-compact, there exist two sequences of compact sets $\{X_m\}$, $\{Y_n\}$, with

$$X_m \subseteq X_{m+1}, \qquad \bigcup_{m=1}^{\infty} X_m = X, \qquad Y_n \subseteq Y_{n+1}, \qquad \bigcup_{n=1}^{\infty} Y_n = Y.$$

By hypothesis, $\mu(X_m), \nu(Y_n) < \infty$ for every m, n.

As usual, we denote by $B_X(x,r)$ the open ball in X with center $x \in X$ and radius $r > 0$. For each $\delta > 0$ consider the open covering $\{B_X(x,\delta/5)\}_{x \in X_m}$ of X_m. Since X_m is a compact set there exists a finite subset $\{x_1,\dots,x_k\} \subseteq X_m$ with

$$X_m \subseteq B_X(x_1,\delta/5) \cup \cdots \cup B_X(x_k,\delta/5).$$

Thus, the measurable sets $\{X_m^i\}_{i=1}^k$ defined as

$$X_m^1 := X_m \cap B_X(x_1,\delta/5),$$
$$X_m^i := X_m \cap B_X(x_i,\delta/5) \setminus \bigcup_{j=1}^{i-1} B_X(x_j,\delta/5), \quad 1 < i \le k,$$

are a partition of X_m and

$$\operatorname{diam}\bigl(X_m^i\bigr) \le \operatorname{diam}\bigl(B_X(x_i, \delta/5)\bigr) < \delta/2$$

for every $1 \le i \le k$.

In a similar way we can find a partition of measurable sets $\{Y_n^j\}_{j=1}^{\ell}$ of Y_n such that $\operatorname{diam}(Y_n^j) < \delta/2$ for every $1 \le j \le \ell$.

Case A. Let us fix m, n and a continuous function $f : X_m \times Y_n \to (0, \infty)$. We are going to prove

$$\frac{1}{\int_{X_m} \frac{d\mu(x)}{\int_{Y_n} f(x,y)\, d\nu(y)}} \ge \int_{Y_n} \frac{d\nu(y)}{\int_{X_m} \frac{d\mu(x)}{f(x,y)}}. \tag{15}$$

Since f is a strictly positive, continuous function on the compact set $X_m \times Y_n$, we have

$$M_{m,n} := \min\bigl\{f(x,y) : (x,y) \in X_m \times Y_n\bigr\} > 0,$$
$$N_{m,n} := \max\bigl\{f(x,y) : (x,y) \in X_m \times Y_n\bigr\} < \infty.$$

Let us fix $\varepsilon > 0$. Since f is uniformly continuous on the compact set $X_m \times Y_n$, there exists $\delta > 0$ such that

$$\bigl|f(x_1,y_1) - f(x_2,y_2)\bigr| < \varepsilon \quad \text{if } d_X(x_1,x_2) + d_Y(y_1,y_2) < \delta.$$

Consider partitions of measurable sets $\{X_m^i\}_{i=1}^k$ of X_m and $\{Y_n^j\}_{j=1}^{\ell}$ of Y_n such that $\operatorname{diam}(X_m^i) < \delta/2$ for every $1 \le i \le k$ and $\operatorname{diam}(Y_n^j) < \delta/2$ for every $1 \le j \le \ell$. If $(x_1,y_1), (x_2,y_2) \in X_m^i \times Y_n^j$, then

$$d_X(x_1,x_2) + d_Y(y_1,y_2) \le \operatorname{diam}\bigl(X_m^i\bigr) + \operatorname{diam}\bigl(Y_n^j\bigr) < \delta$$

and we have $|f(x_1,y_1) - f(x_2,y_2)| < \varepsilon$.

Define

$$M_{m,n}^{i,j} := \inf\bigl\{f(x,y) : (x,y) \in X_m^i \times Y_n^j\bigr\},$$
$$N_{m,n}^{i,j} := \sup\bigl\{f(x,y) : (x,y) \in X_m^i \times Y_n^j\bigr\}.$$

Thus,

$$0 < M_{m,n} \le M_{m,n}^{i,j} \le N_{m,n}^{i,j} \le N_{m,n} < \infty,$$
$$0 \le N_{m,n}^{i,j} - M_{m,n}^{i,j} \le \varepsilon$$

for every $1 \le i \le k$ and $1 \le j \le \ell$.

Let us consider the simple function

$$s_\varepsilon(x,y) := \sum_{i,j} M_{m,n}^{i,j} \chi_{X_m^i \times Y_n^j}(x,y) = \sum_{i,j} M_{m,n}^{i,j} \chi_{X_m^i}(x) \chi_{Y_n^j}(y),$$

where χ_E denotes the characteristic function of the set E, i.e., the function with $\chi_E = 1$ on E and $\chi_E = 0$ on $X \setminus E$.

Since $\{X_m^i \times Y_n^j\}_{i,j}$ is a partition of $X_m \times Y_n$, it is clear that

$$0 < M_{m,n} \le s_\varepsilon \le f \le s_\varepsilon + \varepsilon.$$

We have

$$\int_{Y_n} s_\varepsilon(x,y)\,d\nu(y) = \sum_{i=1}^{k}\left(\sum_{j=1}^{\ell} M_{m,n}^{i,j}\nu\bigl(Y_n^j\bigr)\right)\chi_{X_m^i}(x).$$

Since $\{X_m^i\}_{i=1}^k$ is a partition of X_m, we obtain

$$\begin{aligned}\int_{X_m}\frac{d\mu(x)}{\int_{Y_n} s_\varepsilon(x,y)\,d\nu(y)} &= \int_{X_m}\frac{d\mu(x)}{\sum_{i=1}^k(\sum_{j=1}^\ell M_{m,n}^{i,j}\nu(Y_n^j))\chi_{X_m^i}(x)} \\ &= \int_{X_m}\sum_{i=1}^k \frac{1}{\sum_{j=1}^\ell M_{m,n}^{i,j}\nu(Y_n^j)}\chi_{X_m^i}(x)\,d\mu(x) \\ &= \sum_{i=1}^k \frac{\mu(X_m^i)}{\sum_{j=1}^\ell M_{m,n}^{i,j}\nu(Y_n^j)},\end{aligned}$$

and

$$\begin{aligned}\int_{X_m}\frac{d\mu(x)}{s_\varepsilon(x,y)} &= \int_{X_m}\frac{d\mu(x)}{\sum_{i=1}^k(\sum_{j=1}^\ell M_{m,n}^{i,j}\chi_{Y_n^j}(y))\chi_{X_m^i}(x)} \\ &= \int_{X_m}\sum_{i=1}^k\frac{1}{\sum_{j=1}^\ell M_{m,n}^{i,j}\chi_{Y_n^j}(y)}\chi_{X_m^i}(x)\,d\mu(x) \\ &= \sum_{i=1}^k\frac{\mu(X_m^i)}{\sum_{j=1}^\ell M_{m,n}^{i,j}\chi_{Y_n^j}(y)}.\end{aligned}$$

Since $\{Y_n^j\}_{j=1}^\ell$ is a partition of Y_n, we obtain

$$\begin{aligned}\int_{X_m}\frac{d\mu(x)}{s_\varepsilon(x,y)} &= \sum_{i=1}^k\frac{\mu(X_m^i)}{\sum_{j=1}^\ell M_{m,n}^{i,j}\chi_{Y_n^j}(y)} = \sum_{j=1}^\ell\sum_{i=1}^k\frac{\mu(X_m^i)}{M_{m,n}^{i,j}}\chi_{Y_n^j}(y), \\ \int_{Y_n}\frac{d\nu(y)}{\int_{X_m}\frac{d\mu(x)}{s_\varepsilon(x,y)}} &= \int_{Y_n}\frac{d\nu(y)}{\sum_{j=1}^\ell\sum_{i=1}^k\frac{\mu(X_m^i)}{M_{m,n}^{i,j}}\chi_{Y_n^j}(y)} \\ &= \int_{Y_n}\sum_{j=1}^\ell\frac{1}{\sum_{i=1}^k\frac{\mu(X_m^i)}{M_{m,n}^{i,j}}}\chi_{Y_n^j}(y)\,d\nu(y) = \sum_{j=1}^\ell\frac{\nu(Y_n^j)}{\sum_{i=1}^k\frac{\mu(X_m^i)}{M_{m,n}^{i,j}}}.\end{aligned}$$

Thus, the two following inequalities are equivalent:

$$\frac{1}{\int_{X_m} \frac{d\mu(x)}{\int_{Y_n} s_\varepsilon(x,y)\,d\nu(y)}} \geq \int_{Y_n} \frac{d\nu(y)}{\int_{X_m} \frac{d\mu(x)}{s_\varepsilon(x,y)}},$$

$$\frac{1}{\sum_{i=1}^k \frac{\mu(X_m^i)}{\sum_{j=1}^\ell M_{m,n}^{i,j}\nu(Y_n^j)}} \geq \sum_{j=1}^{\ell} \frac{\nu(Y_n^j)}{\sum_{i=1}^k \frac{\mu(X_m^i)}{M_{m,n}^{i,j}}},$$

and this last inequality holds by Corollary 11, since $M_{m,n}^{i,j} \geq M_{m,n} > 0$, $\sum_{i=1}^k \mu(X_m^i) = \mu(X_m) > 0$ and $\sum_{j=1}^\ell \mu(Y_n^j) = \mu(Y_n) > 0$. Hence, (15) holds for s_ε.

Recall that

$$0 < M_{m,n} \leq s_\varepsilon \leq N_{m,n} \in L^1(X_m \times Y_n, \mu \times \nu),$$

since X_m and Y_n are compact sets and so, $(\mu \times \nu)(X_m \times Y_n) = \mu(X_m)\nu(Y_n) < \infty$. Now, we can choose a sequence of ε converging to 0 and so, the corresponding s_ε converge to f. Since $\mu(X_m), \nu(Y_n) < \infty$, Propositions 12 and 13 give

$$\frac{1}{\int_{X_m} \frac{d\mu(x)}{\int_{Y_n} f(x,y)\,d\nu(y)}} \geq \int_{Y_n} \frac{d\nu(y)}{\int_{X_m} \frac{d\mu(x)}{f(x,y)}}.$$

Case B. Given any $0 < M < N$, consider now any measurable function $f : X_m \times Y_n \to [M,N]$. The hypotheses give that μ, ν are Radon measures. Proposition 19 gives that $\mu \times \nu$ is a Radon measure. Since $X_m \times Y_n$ is a compact set, Proposition 18 gives that there exists a sequence $\{f_k\} \subset C(X_m \times Y_n)$ such that

$$\lim_{k\to\infty} \|f_k - f\|_{L^1(X_m\times Y_n,\mu\times\nu)} = 0.$$

Hence, there exists a subsequence of $\{f_k\}$, that we will denote also by $\{f_k\}$ for simplicity, such that $\lim_{k\to\infty} f_k = f$ $(\mu \times \nu)$-a.e. Let us consider the sequence $\{F_k\}$ defined from $\{f_k\}$ as

$$F_k(x,y) := \begin{cases} f_k(x,y) & \text{if } M \leq f_k(x,y) \leq N, \\ N & \text{if } f_k(x,y) > N, \\ M & \text{if } f_k(x,y) < M. \end{cases}$$

Thus, $\{F_k\} \subset C(X_m \times Y_n)$ and we also have $\lim_{k\to\infty} F_k = f$ $(\mu \times \nu)$-a.e., since $|f - F_k| \leq |f - f_k|$ for every k. We have proved that

$$\frac{1}{\int_{X_m} \frac{d\mu(x)}{\int_{Y_n} F_k(x,y)\,d\nu(y)}} \geq \int_{Y_n} \frac{d\nu(y)}{\int_{X_m} \frac{d\mu(x)}{F_k(x,y)}}$$

for every k. Since $M \leq f, F_k \leq N$ and any constant function belongs to $L^1(X_m \times Y_n, \mu \times \nu)$, Propositions 12 and 13 give

$$\frac{1}{\int_{X_m} \frac{d\mu(x)}{\int_{Y_n} f(x,y)\,d\nu(y)}} \geq \int_{Y_n} \frac{d\nu(y)}{\int_{X_m} \frac{d\mu(x)}{f(x,y)}}.$$

Case C. Consider now any measurable function $f : X_m \times Y_n \to [M, \infty]$, with $M > 0$. If $f_k := \min\{f, k\}$, it is clear that $\lim_{k\to\infty} f_k = f$ and, since

$$0 < \min\{M, 1\} \le f_k \le k,$$

we have proved

$$\frac{1}{\int_{X_m} \frac{d\mu(x)}{\int_{Y_n} f_k(x,y)\,d\nu(y)}} \ge \int_{Y_n} \frac{d\nu(y)}{\int_{X_m} \frac{d\mu(x)}{f_k(x,y)}}$$

for every k. Since $f_k \le f_{k+1}$ for every k and $\mu(X_m) < \infty$, we have

$$\frac{1}{f(x,y)} \le \frac{1}{f_k(x,y)} \le \frac{1}{\min\{M,1\}} \in L^1(X_m, \mu),$$
$$\frac{1}{\int_{Y_n} f(x,y)\,d\nu(y)} \le \frac{1}{\int_{Y_n} f_k(x,y)\,d\nu(y)} \le \frac{1}{\min\{M,1\}\nu(Y_n)} \in L^1(X_m, \mu).$$

Since $f_k \le f_{k+1}$ for every k, Propositions 14 and 16 give

$$\frac{1}{\int_{X_m} \frac{d\mu(x)}{\int_{Y_n} f(x,y)\,d\nu(y)}} \ge \int_{Y_n} \frac{d\nu(y)}{\int_{X_m} \frac{d\mu(x)}{f(x,y)}}.$$

Case D. Consider now any measurable function $f : X_m \times Y_n \to [0, \infty]$. If $f_k := \max\{f, 1/k\}$, it is clear that $\lim_{k\to\infty} f_k = f$ and, since $f_k \ge 1/k$, we have proved

$$\frac{1}{\int_{X_m} \frac{d\mu(x)}{\int_{Y_n} f_k(x,y)\,d\nu(y)}} \ge \int_{Y_n} \frac{d\nu(y)}{\int_{X_m} \frac{d\mu(x)}{f_k(x,y)}}$$

for every k. Since $\mu(X_m), \nu(Y_n) < \infty$ and

$$f(x,y) \le f_1(x,y) \le 1 + f(x,y),$$

$f_1(x,y) \in L^1(Y_n, \nu)$ if and only if $f(x,y) \in L^1(Y_n, \nu)$ for each $x \in X_m$. Since $f_k \ge f_{k+1}$ for every k, Propositions 15 and 17 give

$$\frac{1}{\int_{X_m} \frac{d\mu(x)}{\int_{Y_n} f(x,y)\,d\nu(y)}} \ge \int_{Y_n} \frac{d\nu(y)}{\int_{X_m} \frac{d\mu(x)}{f(x,y)}}.$$

Case E. Fix m and consider any measurable function $f : X_m \times Y \to [0, \infty]$. We have proved for each n

$$\frac{1}{\int_{X_m} \frac{d\mu(x)}{\int_{Y_n} f(x,y)\,d\nu(y)}} \ge \int_{Y_n} \frac{d\nu(y)}{\int_{X_m} \frac{d\mu(x)}{f(x,y)}} = \int_{Y} \chi_{Y_n}(y) \frac{d\nu(y)}{\int_{X_m} \frac{d\mu(x)}{f(x,y)}}.$$

Since $\int_Y f(x,y)\,d\nu(y) \ge \int_{Y_n} f(x,y)\,d\nu(y)$, we have

$$\frac{1}{\int_{X_m} \frac{d\mu(x)}{\int_Y f(x,y)\,d\nu(y)}} \ge \frac{1}{\int_{X_m} \frac{d\mu(x)}{\int_{Y_n} f(x,y)\,d\nu(y)}}.$$

The monotone convergence theorem gives

$$\lim_{n\to\infty}\int_Y \chi_{Y_n}(y)\frac{d\nu(y)}{\int_{X_m}\frac{d\mu(x)}{f(x,y)}} = \int_Y \frac{d\nu(y)}{\int_{X_m}\frac{d\mu(x)}{f(x,y)}}.$$

These three results allow us to conclude that

$$\frac{1}{\int_{X_m}\frac{d\mu(x)}{\int_Y f(x,y)\,d\nu(y)}} \ge \int_Y \frac{d\nu(y)}{\int_{X_m}\frac{d\mu(x)}{f(x,y)}}.$$

Case F. Finally, consider any measurable function $f : X \times Y \to [0,\infty]$. We have proved for each m

$$\frac{1}{\int_X \chi_{X_m}(x)\frac{d\mu(x)}{\int_Y f(x,y)\,d\nu(y)}} = \frac{1}{\int_{X_m}\frac{d\mu(x)}{\int_Y f(x,y)\,d\nu(y)}} \ge \int_Y \frac{d\nu(y)}{\int_{X_m}\frac{d\mu(x)}{f(x,y)}}.$$

Since

$$\int_X \frac{d\mu(x)}{f(x,y)} \ge \int_{X_m}\frac{d\mu(x)}{f(x,y)},$$

we have

$$\int_Y \frac{d\nu(y)}{\int_{X_m}\frac{d\mu(x)}{f(x,y)}} \ge \int_Y \frac{d\nu(y)}{\int_X \frac{d\mu(x)}{f(x,y)}},$$

for each m. The monotone convergence theorem gives

$$\lim_{m\to\infty}\int_X \chi_{X_m}(x)\frac{d\mu(x)}{\int_Y f(x,y)\,d\nu(y)} = \int_X \frac{d\mu(x)}{\int_Y f(x,y)\,d\nu(y)}.$$

These three results give

$$\frac{1}{\int_X \frac{d\mu(x)}{\int_Y f(x,y)\,d\nu(y)}} \ge \int_Y \frac{d\nu(y)}{\int_X \frac{d\mu(x)}{f(x,y)}}.$$

□

Theorem 20 has the following direct consequence for general fractional integrals of Riemann–Liouville type.

Proposition 21 *If $f : [a,b] \times [a,b] \to [0,\infty]$ is a measurable function, then*

$$\frac{1}{\int_a^b \frac{dx}{T(b,x,\alpha)\int_a^b \frac{f(x,y)}{T(b,y,\alpha)}\,dy}} \ge \int_a^b \frac{dy}{T(b,y,\alpha)\int_a^b \frac{dx}{f(x,y)T(b,x,\alpha)}}.$$

5 Conclusions

In this paper we continue with the study and development of an important topic in mathematics that are inequalities, particularly inequalities in a fractional context. The Milne inequality plays an important role in the study of Rosseland's integral for the stellar absorption. In this paper we obtain the Milne-type inequality

$$\frac{1}{\int_X \frac{d\mu(x)}{\int_Y f(x,y)\,d\nu(y)}} \geq \int_Y \frac{d\nu(y)}{\int_X \frac{d\mu(x)}{f(x,y)}},$$

with appropriate hypotheses, and we apply it to the generalized Riemann–Liouville-type integral operators, which include most of the known Riemann–Liouville integral operators.

Although the assumptions of this inequality are not very restrictive, an interesting open problem is to weaken these assumptions for at least one of the two measures.

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