Hindawi
Journal of Mathematics
Volume 2022, Article ID 2247246, 9 pages
https://doi.org/10.1155/2022/2247246

Research Article

On Ostrowski Type Inequalities for Generalized Integral Operators

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Received 28 March 2022; Accepted 2 July 2022; Published 2 August 2022

Academic Editor: Xiaolong Qin

It is well known that mathematical inequalities have played a very important role in solving both theoretical and practical problems. In this paper, we show some new results related to Ostrowski type inequalities for generalized integral operators.

1. Introduction

Mathematical inequalities have been present in the development and consolidation of Science. Nowadays, inequalities are essential tools in multiple applications to different problems, since they are involved in the basis of the processes of approximation, estimation, interpolation, and extrenals and, in general, they appear in the models used in the study of applied problems.

The formalization of mathematical inequalities begins in the 18th century, essentially, with the works of the so-called "Prince of Mathematics" Johann Carl Friedrich Gauss (1777–1855); passing through the investigations and applications of inequalities to Mathematical Analysis developed by Augustin-Louis Cauchy (1789–1857) and Pafnuti Lvóvich Chebyshev (1821–1894). It would be unfair not to mention among the formalizes of mathematical inequalities to Viktor Yakovlevich Bunyakovsky (1804–1889). This remarkable Russian mathematician received all possible mathematical influence from his thesis advisor Augustin-Louis Cauchy. This remarkable scientist is credited with having proved in 1859, many years before Hermann Schwarz, the well-known Cauchy–Schwarz Inequality for the infinite-dimensional case. It is worth noting that in many texts the famous inequality is known as: Cauchy–Bunyakovsky–Schwarz.

The proof of Hardy's famous inequality involved an important group of prominent mathematicians of his time such as Edmund Hermann Landau (1887–1938), George Pólya (1887–1985), and Issai Schur (1875–1941), and Marcel Riesz (1886–1969), among others. It is worth noting the coordinating role played by Godfrey Harold Hardy (1887–1947) in the study of inequalities; his work has been very significant, fundamentally, for the systematization and application of the Theory of Mathematical Inequalities. Hardy was the founder of the Journal of the London Mathematical Society, a suitable publication for many articles on inequalities. In addition, along with Littlewood and Polya, Hardy was the editor of the volume Inequalities see [1], which was the first monograph, on inequalities and immediately used as the basis for the later development of mathematical inequalities. For more information on the epistemological evolution of the Theory of Mathematical Inequalities see [2].

In recent years there has been a growing interest in the study of many classical inequalities applied to integral operators associated with different types of fractional derivatives since integral inequalities and their applications play a vital role in the theory of differential equations and applied mathematics. For the most recent works, we recommend the reader [3–5]. Some of the inequalities studied are Gronwall, Chebyshev, Hermite–Hadamard-type, Ostrowski-type, Grüss-type, Hardy-type, Gagliardo–Nirenberg-type, reverse Minkowski and reverse Hölder inequalities (see, e.g., [6–13]). Other types of inequalities such as Hilber-type inequalities and Hadamard-type inequalities have also been recently studied in the context of integral equations and generalized mapping on fractal sets, [14, 15].

In this paper, we show some new results related to Ostrowski-type inequalities via conformable and non-conformable operators.

Ostrowski-type inequalities have significant contributions to the area of numerical analysis since they provide estimates of the error of many quadrature rules, for example, the midpoint rule, Simpson's rule, the trapezoidal rule, and other generalized fractional integrals. They also have many powerful results and a large number of applications in Probability Theory and Statistics, Information Theory, and Integral Operator Theory. For further discussions, we refer the reader to the book by Dragomir and Rassias (see [16]).

2. On Generalized Ostrowski's Inequality

Alexander Ostrowski (1893–1986) was an important mathematician born in Kyiv, the former Russian empire, today the capital of Ukraine. From a mathematical point of view, he was directly influenced by great mathematicians such as Hensel, Hilbert, Klein, and Landau. Another tribute, in addition to the inequality that bears his name, is the well-known Ostrowski Prize that is jointly sponsored by several renowned universities and the Academies of Science of the Netherlands and Denmark.

Ostrowski proved in [17] an integral inequality associated with a real differentiable function that establishes an upper bound for the difference between the function evaluated at any interior point of some interval and the average of the function over the same interval.

Theorem 1. *Let $f\colon [a,b] \longrightarrow \mathbb{R}$ be a continuous function which is differentiable on (a,b). If $f' \in L^{\infty}[a,b]$, then*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t \right| \leq \frac{1}{b-a}\left[\left(\frac{b-a}{2}\right)^2 + \left(x - \frac{a+b}{2}\right)^2\right]\|f'\|_{\infty}. \tag{1}$$

Since then, there are a lot of generalizations and applications of this inequality (see, e.g., [16]). In particular, Dragomir and Wang generalized this inequality to $L^p[a,b]\,(p>1)$ in [18] as follows:

Theorem 2. *Let $f\colon [a,b] \longrightarrow \mathbb{R}$ be a continuous function which is differentiable on (a,b). If $p>1$, $(1/p)+(1/q)=1$ and $f' \in L^p[a,b]$, then*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t \right| \leq \left[\frac{(x-a)^{q+1} + (b-x)^{q+1}}{(q+1)(b-a)^q}\right]^{1/q}\|f'\|_p. \tag{2}$$

In this paper, we prove the following two weighted versions of this inequality. The main improvement is to consider general weights, but also, we prove the inequality for a larger class of functions, and we include the case $p=1$. Furthermore, we prove that our inequality is sharp for every weight.

Theorem 3. *Let $f\colon [a,b] \longrightarrow \mathbb{R}$ be an absolutely continuous function, and $w\colon [a,b] \longrightarrow [0,\infty)$ an integrable function with $\int_a^b w(t)dt > 0$.*

(1) If $1 < p \leq \infty$ and $(1/p)+(1/q)=1$, then

$$\left| f(x) - \frac{1}{\int_a^b w(t)\mathrm{d}t}\int_a^b f(t)w(t)\mathrm{d}t \right| \leq \left(\frac{b-a}{2} + \left|x - \frac{a+b}{2}\right|\right)^{1/q}\|f'\|_p. \tag{3}$$

(2) If $p=1$, then

$$\left| f(x) - \frac{1}{\int_a^b w(t)\mathrm{d}t}\int_a^b f(t)w(t)\mathrm{d}t \right| \leq \|f'\|_1. \tag{4}$$

Theorem 3 provides simple bounds, but they do not depend on the weight w. This theorem can be improved by the following bounds involving the weight.

Theorem 4. *Let $f\colon [a,b] \longrightarrow \mathbb{R}$ be an absolutely continuous function, and $w\colon [a,b] \longrightarrow [0,\infty)$ an integrable function with $\int_a^b w(t)dt > 0$.*

(1) If $1 < p \leq \infty$ and $(1/p)+(1/q)=1$, then

$$\left| f(x) - \frac{1}{\int_a^b w(t)\mathrm{d}t}\int_a^b f(t)w(t)\mathrm{d}t \right| \leq \frac{1}{\int_a^b w(t)\mathrm{d}t}\left(\int_a^x\left(\int_a^t w(s)\mathrm{d}s\right)^q \mathrm{d}t + \int_x^b\left(\int_t^b w(s)\mathrm{d}s\right)^q \mathrm{d}t\right)^{1/q}\|f'\|_p. \tag{5}$$

(2) If $p=1$, then

$$\left| f(x) - \frac{1}{\int_a^b w}\int_a^b f(t)w(t)\mathrm{d}t \right| \leq \frac{1}{\int_a^b w(t)\mathrm{d}t}\max\left\{\int_a^x w(t)\mathrm{d}t, \int_x^b w(t)\mathrm{d}t\right\}\|f'\|_1. \tag{6}$$

(3) For each weight w, $1 < p < \infty$ and $x \in [a,b]$, there exists an absolutely continuous function f with $f' \in L^p[a,b]$ such that the equality in the inequality holds.

3. Proofs of the Inequalities

Recall that a function $f\colon [a,b] \longrightarrow \mathbb{R}$ is *absolutely continuous* if for every $\varepsilon > 0$, there exists $\delta > 0$ such that if a finite sequence of pairwise disjoint sub-intervals $\{[x_k, y_k]\}_k \subset [a,b]$ satisfies

$$\sum_k (y_k - x_k) < \delta, \tag{7}$$

then

$$\sum_k \left| f(y_k) - f(x_k) \right| < \varepsilon. \tag{8}$$

It is well-known that f is absolutely continuous on $[a,b]$ if and only if it has a derivative f' almost everywhere, the derivative is integrable, and $f(x) = f(a) + \int_a^x f'(t)\mathrm{d}t$ for every $x \in [a,b]$.

Proof of Theorem 3. First of *all*, note that $fw \in L^1[a,b]$, since $f \in L^\infty[a,b]$ and $w \in L^1[a,b]$. We can assume that $f' \in L^p[a,b]$, since otherwise the inequality trivially holds.

Let us define m and M as the minimum and maximum values of f on $[a,b]$, respectively. Thus, we have

$$m \le \frac{1}{\int_a^b w(t)\mathrm{d}t} \int_a^b f(t)w(t)\mathrm{d}t \le M. \tag{9}$$

The intermediate values theorem gives that there exists $x_0 \in [a,b]$ with

$$f(x_0) = \frac{1}{\int_a^b w(t)\mathrm{d}t} \int_a^b f(t)w(t)\mathrm{d}t. \tag{10}$$

Assume first $1 < p < \infty$. Hölder inequality gives

$$\begin{aligned} \left| f(x) - \frac{1}{\int_a^b w(t)\mathrm{d}t} \int_a^b f(t)w(t)\mathrm{d}t \right| &= \left| f(x) - f(x_0) \right| = \left| \int_{x_0}^x f'(t)\mathrm{d}t \right| \\ \le \left| \int_{x_0}^x |f'(t)|^p \mathrm{d}t \right|^{1/p} \left| \int_{x_0}^x 1^q \mathrm{d}t \right|^{1/q} &= |x - x_0|^{1/q} \left| \int_{x_0}^x |f'(t)|^p \mathrm{d}t \right|^{1/p} \\ &\le \max\{x - a, b - x\}^{1/q} \|f'\|_p. \end{aligned} \tag{11}$$

The desired inequality holds since

$$\max\{x - a, b - x\} = \frac{b-a}{2} + \left| x - \frac{a+b}{2} \right|. \tag{12}$$

If $p = 1$ or $p = \infty$, then a similar and simpler argument gives the inequalities. □

Proof of Theorem 4. We can assume that $f' \in L^p[a,b]$, since otherwise the inequality trivially holds. Since $w \in L^1[a,b]$, the function $\int_A^t w(s)\mathrm{d}s$ is absolutely continuous on $[a,b]$ for every $A \in [a,b]$.

Since the integration by parts rule holds for absolutely continuous functions, we have

$$\begin{aligned} \int_a^x \left(\int_a^t w(s)\mathrm{d}s \right) f'(t)\mathrm{d}t &= \left[f(t) \int_a^t w(s)\mathrm{d}s \right]_a^x - \int_a^x f(t)w(t)\mathrm{d}t \\ &= f(x) \int_a^x w(t)\mathrm{d}t - \int_a^x f(t)w(t)\mathrm{d}t, \\ \int_x^b \left(\int_b^t w(s)\mathrm{d}s \right) f'(t)\mathrm{d}t &= \left[f(t) \int_b^t w(s)\mathrm{d}s \right]_x^b - \int_x^b f(t)w(t)\mathrm{d}t \\ &= f(x) \int_x^b w(t)\mathrm{d}t - \int_x^b f(t)w(t)\mathrm{d}t, \end{aligned} \tag{13}$$

and so,

$$\begin{aligned} &f(x) \int_a^b w(t)\mathrm{d}t - \int_a^b f(t)w(t)\mathrm{d}t \\ &\quad = \int_a^x \left(\int_a^t w(s)\mathrm{d}s \right) + \int_x^b \left(\int_b^t w(s)\mathrm{d}s \right) f'(t)\mathrm{d}t. \end{aligned} \tag{14}$$

Assume first $1 < p < \infty$. Hölder inequality gives

$$\begin{aligned} &\left| \int_a^x \left(\int_a^t w(s)\mathrm{d}s \right) f'(t)\mathrm{d}t \right| + \left| \int_x^b \left(\int_t^b w(s)\mathrm{d}s \right) f'(t)\mathrm{d}t \right| \\ &\le \left(\int_a^x |f'(t)|^p \mathrm{d}t \right)^{1/p} \left(\int_a^x \left(\int_a^t w(s)\mathrm{d}s \right)^q \mathrm{d}t \right)^{1/q} \\ &+ \left(\int_x^b |f'(t)|^p \mathrm{d}t \right)^{1/p} \left(\int_x^b \left(\int_t^b w(s)\mathrm{d}s \right)^q \mathrm{d}t \right)^{1/q}, \end{aligned} \tag{15}$$

Thus, discrete Hölder inequality $a_1 b_1 + a_2 b_2 \le (a_1^p + a_2^p)^{1/p} (b_1^q + b_2^q)^{1/q}$ gives

$$\begin{aligned} &\left| \int_a^x \left(\int_a^t w(s)\mathrm{d}s \right) f'(t)\mathrm{d}t \right| + \left| \int_x^b \left(\int_t^b w(s)\mathrm{d}s \right) f'(t)\mathrm{d}t \right| \\ &\le \left(\int_a^x |f'(t)|^p \mathrm{d}t \right)^{1/p} \left(\int_a^x \left(\int_a^t w(s)\mathrm{d}s \right)^q \mathrm{d}t \right)^{1/q} \\ &+ \left(\int_x^b |f'(t)|^p \mathrm{d}t \right)^{1/p} \left(\int_x^b \left(\int_t^b w(s)\mathrm{d}s \right)^q \mathrm{d}t \right)^{1/q} \\ &\le \left(\int_a^x |f'(t)|^p \mathrm{d}t + \int_x^b |f'(t)|^p \mathrm{d}t \right)^{1/p} \\ &\cdot \left(\int_a^x \left(\int_a^t w(s)\mathrm{d}s \right)^q \mathrm{d}t + \int_x^b \left(\int_t^b w(s)\mathrm{d}s \right)^q \mathrm{d}t \right)^{1/q} \\ &= \left(\int_a^x \left(\int_a^t w(s)\mathrm{d}s \right)^q \mathrm{d}t + \int_x^b \left(\int_t^b w(s)\mathrm{d}s \right)^q \mathrm{d}t \right)^{1/q} \|f'\|^p, \end{aligned} \tag{16}$$

and the inequality holds.

Assume now that $p = 1$. Note that, since $w \ge 0$,

$$\begin{aligned}
&\left|\int_a^x\left(\int_a^t w(s)\mathrm{d}s\right)f'(t)\mathrm{d}t\right| + \left|\int_x^b\left(\int_t^b w(s)\mathrm{d}s\right)f'(t)\mathrm{d}t\right| \\
&\quad\leq \left(\int_a^x w(s)\mathrm{d}s\right)\int_a^x\left|f'(t)\right|\mathrm{d}t + \left(\int_x^b w(s)\mathrm{d}s\right)\int_x^b\left|f'(t)\right|\mathrm{d}t \\
&\quad\leq \max\left\{\int_a^x w(t)\mathrm{d}t, \int_x^b w(t)\mathrm{d}t\right\}\int_a^x\left|f'(t)\right|\mathrm{d}t + \max\left\{\int_a^x w(t)\mathrm{d}t, \int_x^b w(t)\mathrm{d}t\right\}\int_x^b\left|f'(t)\right|\mathrm{d}t \\
&\quad= \max\left\{\int_a^x w(t)\mathrm{d}t, \int_x^b w(t)\mathrm{d}t\right\}\left\|f'\right\|_1,
\end{aligned} \tag{17}$$

and the inequality also holds in this case.

If $p=\infty$, then a similar argument gives the inequality.

Finally, let us prove (3). Fix w, $1<p<\infty$ and $x\in[a,b]$, and define

$$f(t) = \begin{cases} \displaystyle\int_x^t\left(\int_a^s w(\tau)d\tau\right)^{1/(p-1)}\mathrm{d}s, & \text{if } t\in[a,x], \\ \displaystyle -\int_x^t\left(\int_s^b w(\tau)d\tau\right)^{1/(p-1)}\mathrm{d}s, & \text{if } t\in[x,b]. \end{cases} \tag{18}$$

Since $w\geq 0$ and $w\in L^1[a,b]$, we have

$$\%\left(\int_a^s w(\tau)\mathrm{d}\tau\right)^{1/(p-1)} \in C[a,x], \left(\int_s^b w(\tau)\mathrm{d}\tau\right)^{1/(p-1)} \in C[x,b], \tag{19}$$

and so, f is an absolutely continuous function on $[a,b]$ and $f'\in L^{\infty}[a,b]$.

The argument in the proof of item (1) shows that it suffices to check that

$$\begin{aligned}
&\int_a^x\left(\int_a^t w(s)\mathrm{d}s\right)f'(t)\mathrm{d}t + \int_x^b\left(\int_b^t w(s)\mathrm{d}s\right)f'(t)\mathrm{d}t \\
&\quad= \left(\int_a^x\left(\int_a^t w(s)\mathrm{d}s\right)^q\mathrm{d}t + \int_x^b\left(\int_t^b w(s)\mathrm{d}s\right)^q\mathrm{d}t\right)^{1/q}\left\|f'\right\|_p.
\end{aligned} \tag{20}$$

Note that,

$$\begin{aligned}
&\int_a^x\left(\int_a^t w(s)\mathrm{d}s\right)f'(t)\mathrm{d}t + \int_x^b\left(\int_b^t w(s)\mathrm{d}s\right)f'(t)\mathrm{d}t \\
&\quad= \int_a^x\left(\int_a^t w(s)\mathrm{d}s\right)\left(\int_a^t w(s)\mathrm{d}s\right)^{1/(p-1)}\mathrm{d}t \\
&\qquad - \int_x^b\left(\int_b^t w(s)\mathrm{d}s\right)\left(\int_t^b w(s)\mathrm{d}s\right)^{1/(p-1)}\mathrm{d}t \\
&\quad= \int_a^x\left(\int_a^t w(s)\mathrm{d}s\right)^{p/(p-1)}\mathrm{d}t + \int_x^b\left(\int_t^b w(s)\mathrm{d}s\right)^{p/(p-1)}\mathrm{d}t \\
&\qquad \cdot\left(\int_a^x\left(\int_a^t w(s)\mathrm{d}s\right)^q\mathrm{d}t + \int_x^b\left(\int_t^b w(s)\mathrm{d}s\right)^q\mathrm{d}t\right)^{1/q}\left\|f'\right\|_p \\
&\quad= \left(\int_a^x\left(\int_a^t w(s)\mathrm{d}s\right)^{p/(p-1)}\mathrm{d}t + \int_x^b\left(\int_t^b w(s)\mathrm{d}s\right)^{p/(p-1)}\mathrm{d}t\right)^{(p-1)/p} \\
&\qquad \cdot\left(\int_a^x\left(\int_a^t w(s)\mathrm{d}s\right)^{p/(p-1)}\mathrm{d}t + \int_x^b\left(\int_t^b w(s)\mathrm{d}s\right)^{p/(p-1)}\mathrm{d}t\right)^{1/p} \\
&\quad= \int_a^x\left(\int_a^t w(s)\mathrm{d}s\right)^{p/(p-1)}\mathrm{d}t + \int_x^b\left(\int_t^b w(s)\mathrm{d}s\right)^{p/(p-1)}\mathrm{d}t,
\end{aligned} \tag{21}$$

and so, the equality in the inequality in the first item is attained for this choice of f.

Note that, if we substitute the weight w by the constant function 1 in Theorem 4, then we get the classical inequality described in Theorem 2. □

4. On the Ostrowski Inequality in Conformable and Nonconformable Context

The evolution of many physical processes can be described in a more precise way by using fractional derivatives (see, e.g., [19–24]). Usually, it suffices to replace the time derivative in a given evolution equation by a fractional derivative. There is a solid mathematical basis for proceeding this way (see, e.g., [23–26]). Recent developments on fractional calculus and its applications can be found in [27–30].

In several papers (see, e.g., [23, 31–33]) are defined local fractional derivatives in the following way. Given a function $f(t)$, $\alpha \in (0,1]$ and a kernel $T(t,\alpha)$, the derivative of f of order α at the point t with respect to the kernel T is defined by the following equation:

$$G_T^{\alpha} f(t) = \lim_{h \longrightarrow 0} \frac{f(t) - f(t - hT(t,\alpha))}{h}. \quad (22)$$

Let I be an (open or not) interval $I \subseteq \mathbb{R}$. The generalized derivative G_T^{α} is said *conformable* if $G_T^1 f(t) = f'(t)$ for every $t \in I$ or, equivalently, if $T(t,1) = 1$ for every $t \in I$.

Let I be an interval $I \subseteq \mathbb{R}$, $a, t \in I$, $\alpha \in (0,1]$ and T a positive continuous function on $I \times (0,1]$. In [33] the integral operator $J_{T,a}^{\alpha}$ is defined for every locally integrable function f on I as follows:

$$J_{T,a}^{\alpha}(f)(t) = \int_a^t \frac{f(s)}{T(s,\alpha)} ds. \quad (23)$$

Also, the integral operator $J_{T,a}^{\alpha}$ is said *conformable* if $T(t,1) = 1$ for every $t \in I$.

The following basic properties related to the operator G_T^{α} appear in [34].

Theorem 5 (see [34], Theorem 2.4). *Let I be an interval* $I \subseteq \mathbb{R}$, $f: I \longrightarrow \mathbb{R}$ *and* $\alpha \in \mathbb{R}^+$.

(1) If there exists $D^{\lceil \alpha \rceil} f$ *at the point* $t \in I$ *, then* f *is* G_T^{α} *-differentiable at* t *and* $G_T^{\alpha} f(t) = T(t,\alpha)^{\lceil \alpha \rceil} D^{\lceil \alpha \rceil} f(t)$.

(2) If $\alpha \in (0,1]$ *, then* f *is* G_T^{α} *-differentiable at* $t \in I$ *if and only if* f *is differentiable at* t *; in this case, we have* $G_T^{\alpha} f(t) = T(t,\alpha) f'(t)$.

Theorem 6 (see [34] Theorem 2.5). *Let I be an interval* $I \subseteq \mathbb{R}$, $f, g: I \longrightarrow \mathbb{R}$ *and* $\alpha \in \mathbb{R}^+$. *Assume that* f, g *are* G_T^{α} *-differentiable functions at* $t \in I$ *. Then the following statements hold:*

(1) $af + bg$ *is* G_T^{α}*-differentiable at* t *for every* $a, b \in \mathbb{R}$, *and*$G_T^{\alpha}(af + bg)(t) = aG_T^{\alpha} f(t) + bG_T^{\alpha} g(t)$

(2) If$\alpha \in (0,1]$, *then*fg *is*G_T^{α}*-differentiable at*t *and*$G_T^{\alpha}(fg)(t) = f(t)G_T^{\alpha} g(t) + g(t)G_T^{\alpha} f(t)$

(3) If$\alpha \in (0,1]$ *and*$g(t) \neq 0$, *then*f/g *is*G_T^{α}*-differentiable at*t *and*$G_T^{\alpha(f/g)(t)=(g(t)<>G_T^{\alpha} f(t)-}$ $f(t) <> G_T^{\alpha} g(t)/g(t)^2)$

(4) $G_T^{\alpha}(\lambda) = 0$, *for every* $\lambda \in \mathbb{R}$

Theorem 7 (see [34] Theorem 2.10). *(Let*$\alpha \in (0,1]$,g *a* G_T^{α}*-differentiable function at*t *and*f *a differentiable function at*$g(t)$. *Then*$f^{\circ}g$ *is*G_T^{α}*-differentiable at*t, *and*$G_T^{\alpha}(f^{\circ}g)(t) = f'(g(t)) <> G_T^{\alpha} g(t)$.

To review a good summary of some elementary properties associated with the integral operator $J_{T,a}^{\alpha}$, we recommend reading the paper [33].

Theorem 8 (see [33] Theorem 8). *Let I be an interval* $I \subseteq \mathbb{R}$, $a, b \in I$ *and* $\alpha \in \mathbb{R}$. *Suppose that* f, g *are locally integrable functions on* I , *and* $k_1, k_2 \in \mathbb{R}$. *Then, we have*

(1) $J_{T,a}^{\alpha}(k_1 f + k_2 g)(t) = k_1 J_{T,a}^{\alpha} f(t) + k_2 J_{T,a}^{\alpha} g(t)$

(2) if $f \geq g$, *then* $J_{T,a}^{\alpha} f(t) \geq J_{T,a}^{\alpha} g(t)$ *for every* $t \in I$ *with* $t \geq a$

(3) $|J_{T,a}^{\alpha} f(t)| \leq J_{T,a}^{\alpha} |f|(t)$ *for every* $t \in I$ *with* $t \geq a$

(4) $\int_a^b (f(s)/T(s,\alpha)) ds = J_{T,a}^{\alpha} f(t) - J_{T,b}^{\alpha} f(t) = J_{T,a}^{\alpha} f(t)(b)$ *for every* $t \in I$

Proposition 1 (see [33] Proposition 6). *Let I be an interval* $I \subseteq \mathbb{R}$, $a \in I$, $\alpha \in (0,1]$, T *a positive continuous function on* $I \times (0,1]$, *and* f *a differentiable function on I such that* f' *is a locally integrable function on* I . *Then, we have for all* $t \in I$

$$J_{T,a}^{\alpha}\left(G_T^{\alpha}(f)\right)(t) = f(t) - f(a). \quad (24)$$

In [23] it is defined the integral operator $J_{T,a}^{\alpha}$ for the specific choice of the kernel T given by $T(t,\alpha) = t^{1-\alpha}$, and ([23], Theorem 3.1) shows

$$G_{t^{1-\alpha}}^{\alpha}\left(J_{t^{1-\alpha},a}^{\alpha}(f)\right)(t) = f(t), \quad (25)$$

for every continuous function f on I, $a, t \in I$ and $\alpha \in (0,1]$. Hence, Proposition 2 below extends to any T this important equality (see [33]).

Proposition 2 (see [33] Proposition 7). *Let I be an interval* $I \subseteq \mathbb{R}$, $a \in I$, $\alpha \in (0,1]$ *and* T *a positive continuous function on* $I \times (0,1]$. *Then,*

$$G_T^{\alpha}\left(J_{T,a}^{\alpha}(f)\right)(t) = f(t), \quad (26)$$

for every continuous function f *on* I *and* $a, t \in I$.

For further information about this integral operator and its applications, we refer the readers to [25, 33–35].

If we take $w(t) = (1/T)(t,\alpha)$ in Theorem 4 with $I = [a,b]$, we can obtain inequalities involving integral operators, conformable with $T(t,\alpha) = (t-a)^{1-\alpha}$ (see Proposition 3) and nonconformable with $T(t,\alpha) = e^{-\alpha(t-a)}$ (see Propositions 4 and 5).

Recall that the *incomplete beta functions*B_1 and B_2 are defined, respectively, as follows:

$$B_1(x;u,v) = \int_0^x t^{u-1}(1-t)^{v-1}\mathrm{d}t, \tag{27}$$

for $x \in [0,1)$, $u > 0$, and

$$B_2(x;u,v) = \int_x^1 t^{u-1}(1-t)^{v-1}\mathrm{d}t, \tag{28}$$

for $x \in (0,1]$, $v > 0$.

Proposition 3. *Let $f\colon [a,b] \longrightarrow \mathbb{R}$ be an absolutely continuous function, and $\alpha > 0$.*

(1) If $1 < p \le \infty$ and $(1/p) + (1/q) = 1$, then

$$\begin{aligned}&\left|f(x) - \frac{\alpha}{(b-a)^\alpha}\int_a^b f(t)(t-a)^{\alpha-1}\mathrm{d}t\right| \\ &\quad\le \frac{\alpha}{(b-a)^\alpha}\left(\frac{(x-a)^{\alpha q+1}}{\alpha^q(\alpha q+1)} + \frac{(b-a)^{\alpha q+1}}{\alpha^{q+1}}B_1\left(1-\left(\frac{x-a}{b-a}\right)^\alpha; q+1, \frac{1}{\alpha}\right)\right)^{1/q}\|f'\|_p.\end{aligned} \tag{29}$$

(2) If $p = 1$, then

$$\begin{aligned}&\left|f(x) - \frac{\alpha}{(b-a)^\alpha}\int_a^b f(t)(t-a)^{\alpha-1}\mathrm{d}t\right| \\ &\quad\le \frac{1}{(b-a)^\alpha}\max\left\{(x-a)^\alpha, (b-a)^\alpha - (x-a)^\alpha\right\}\|f'\|_1.\end{aligned} \tag{30}$$

Proof. Note that $w(x) = (x-a)^{\alpha-1} \in L^1[a,b]$, since $\alpha > 0$. We have

$$\begin{aligned}&\int_a^b w(t)\mathrm{d}t = \frac{(b-a)^\alpha}{\alpha}, \\ &\int_a^x\left(\int_a^t w(s)\mathrm{d}s\right)^q \mathrm{d}t = \int_a^x \frac{(t-a)^{\alpha q}}{\alpha^q}\mathrm{d}t = \frac{(x-a)^{\alpha q+1}}{\alpha^q(\alpha q+1)}.\end{aligned} \tag{31}$$

By making the change of variables $u^{1/\alpha} = (t-a)/(b-a)$, we obtain

$$\begin{aligned}\int_x^b\left(\int_t^b w(s)\mathrm{d}s\right)^q \mathrm{d}t &= \int_x^b \frac{((b-a)^\alpha - (t-a)^\alpha)^q}{\alpha^q}\mathrm{d}t \\ &= \frac{(b-a)^{\alpha q+1}}{\alpha^{q+1}}\int_{((x-a)/(b-a))^\alpha}^1 (1-u)^q u^{(1-t\alpha)/\alpha}\mathrm{d}u \\ &= \frac{(b-a)^{\alpha q+1}}{\alpha^{q+1}}\int_0^{1-((x-a)/(b-a))^\alpha} t^{q+1-1}(1-t)^{1/(\alpha-1)}\mathrm{d}t \\ &= \frac{(b-a)^{\alpha q+1}}{\alpha^{q+1}}B_1\left(1-\left(\frac{x-a}{b-a}\right)^\alpha; q+1, \frac{1}{\alpha}\right),\end{aligned} \tag{32}$$

thus, Theorem 4 gives the inequalities. □ □

Proposition 4. *Let $f\colon [a,b] \longrightarrow \mathbb{R}$ be an absolutely continuous function, and $\alpha > 0$.*

(1) If $1 < p \le \infty$ and $(1/p) + (1/q) = 1$, then

$$\begin{aligned}&\left|f(x) - \frac{\alpha}{e^{\alpha(b-a)}-1}\int_a^b f(t)e^{\alpha(t-a)}\mathrm{d}t\right| \\ &\quad\le \frac{\alpha^{-q}}{e^{\alpha(b-a)}-1}\left(B_2\left(e^{-\alpha(x-a)}; -q, q+1\right) + e^{\alpha q(b-a)}B_2\left(e^{\alpha(x-b)}; 0, q+1\right)\right)\|f'\|_p.\end{aligned} \tag{33}$$

(2) If $p = 1$, then

$$\begin{aligned}&\left|f(x) - \frac{\alpha}{e^{\alpha(b-a)}-1}\int_a^b f(t)e^{\alpha(t-a)}\mathrm{d}t\right| \\ &\quad\le \frac{1}{e^{\alpha(b-a)}-1}\max\left\{e^{\alpha(x-a)} - 1, e^{\alpha(b-a)} - e^{\alpha(x-a)}\right\}\|f'\|_1.\end{aligned} \tag{34}$$

Proof. Note that $w(x) = e^{\alpha(x-a)} \in L^1[a,b]$. We have

$$\int_a^b w(t)\mathrm{d}t = \frac{e^{\alpha(b-a)}-1}{\alpha}. \tag{35}$$

By making the change of variables $u = e^{-\alpha(t-a)}$, we obtain

$$\int_a^x \left(\int_a^t w(s)\mathrm{d}s\right)^q \mathrm{d}t = \int_a^x \frac{\left(e^{\alpha(t-a)}-1\right)^q}{\alpha^q}\mathrm{d}t$$
$$= \int_a^x \frac{e^{\alpha(q+1)(t-a)}\left(1-e^{-\alpha(t-a)}\right)^q}{\alpha^{q+1}}\alpha e^{-\alpha(t-a)}\mathrm{d}t$$
$$= \frac{1}{\alpha^{q+1}}\int_{e^{-\alpha(x-a)}}^1 u^{-q-1}(1-u)^q\mathrm{d}u$$
$$= \frac{1}{\alpha^{q+1}}B_2\left(e^{-\alpha(x-a)};-q,q+1\right). \tag{36}$$

By making the change of variables $u=e^{\alpha(t-b)}$, we have

$$\int_x^b \left(\int_t^b w(s)\mathrm{d}s\right)^q \mathrm{d}t = \int_x^b \frac{\left(e^{\alpha(b-a)}-e^{\alpha(t-a)}\right)^q}{\alpha^q}\mathrm{d}t$$
$$= \frac{e^{\alpha q(b-a)}}{\alpha^{q+1}}\int_x^b \left(1-e^{\alpha(t-b)}\right)^q e^{-\alpha(t-b)}\alpha e^{\alpha(t-b)}\mathrm{d}t$$
$$= \frac{e^{\alpha q(b-a)}}{\alpha^{q+1}}\int_{e^{\alpha(x-tb)}}^1 (1-u)^q u^{-1}\mathrm{d}u$$
$$= \frac{e^{\alpha q(b-a)}}{\alpha^{q+1}}B_2\left(e^{\alpha(x-b)};0,q+1\right). \tag{37}$$

Therefore, Theorem 4 gives the inequalities. □ □

Proposition 5. *Let $f\colon [a,b]\longrightarrow \mathbb{R}$ be an absolutely continuous function, and $\alpha<0$.*

(1) If $1<p\leq\infty$ and $(1/p)+(1/q)=1$, then

$$\left|f(x)-\frac{|\alpha|}{1-e^{\alpha(b-a)}}\int_a^b f(t)e^{\alpha(t-a)}\mathrm{d}t\right|$$
$$\leq \frac{|\alpha|^{-q}}{1-e^{\alpha(b-a)}}\left(B_2\left(e^{\alpha(x-a)};0,q+1\right)\right. \tag{38}$$
$$\left.+e^{\alpha q(b-a)}B_2\left(e^{-\alpha(x-b)};-q,q+1\right)\right)\|f'\|_p.$$

(2) If $p=1$, then

$$\left|f(x)-\frac{|\alpha|}{1-e^{\alpha(b-a)}}\int_a^b f(t)e^{\alpha(t-a)}\mathrm{d}t\right|$$
$$\leq \frac{1}{1-e^{\alpha(b-a)}}\max\left\{1-e^{\alpha(x-a)},e^{\alpha(x-a)}-e^{\alpha(b-a)}\right\}\|f'\|_1. \tag{39}$$

Proof. Note that, $w(x)=e^{\alpha(x-a)}\in L^1[a,b]$. We have

$$\int_a^b w(t)\mathrm{d}t = \frac{e^{\alpha(b-a)}-1}{\alpha} = \frac{1-e^{\alpha(b-a)}}{|\alpha|}. \tag{40}$$

By making the change of variables $u=e^{\alpha(t-a)}$, we obtain

$$\int_a^x \left(\int_a^t w(s)\mathrm{d}s\right)^q \mathrm{d}t = \int_a^x \frac{\left(1-e^{\alpha(t-a)}\right)^q}{|\alpha|^q}\mathrm{d}t$$
$$= \int_a^x \frac{e^{-\alpha(t-a)}\left(1-e^{\alpha(t-a)}\right)^q}{|\alpha|^{q+1}}(-\alpha)e^{\alpha(t-a)}\mathrm{d}t$$
$$= \frac{1}{|\alpha|^{q+1}}\int_{e^{\alpha(x-ta)}}^1 u^{-1}(1-u)^q\mathrm{d}u$$
$$= \frac{1}{|\alpha|^{q+1}}B_2\left(e^{\alpha(x-a)};0,q+1\right). \tag{41}$$

By making the change of variables $u=e^{-\alpha(t-b)}$, we have

$$\int_x^b \left(\int_t^b w(s)\mathrm{d}s\right)^q \mathrm{d}t = \int_x^b \frac{\left(e^{\alpha(t-a)}-e^{\alpha(b-a)}\right)^q}{|\alpha|^q}\mathrm{d}t$$
$$= \frac{1}{|\alpha|^{q+1}}\int_x^b e^{\alpha q(t-a)}\left(1-e^{-\alpha(t-b)}\right)^q e^{\alpha(t-b)}$$
$$(-\alpha)e^{-\alpha(t-b)}\mathrm{d}t$$
$$= \frac{e^{\alpha q(b-a)}}{|\alpha|^{q+1}}\int_x^b e^{\alpha q(t-b)}\left(1-e^{-\alpha(t-b)}\right)^q e^{\alpha(t-b)}$$
$$(-\alpha)e^{-\alpha(t-b)}\mathrm{d}t$$
$$= \frac{e^{\alpha q(b-a)}}{|\alpha|^{q+1}}\int_{e^{-\alpha(x-b)}}^1 u^{-q-1}(1-u)^q\mathrm{d}u$$
$$= \frac{e^{\alpha q(b-a)}}{|\alpha|^{q+1}}B_2\left(e^{-\alpha(x-b)};-q,q+1\right), \tag{42}$$

thus, Theorem 4 gives the inequalities. □ □

5. Conclusions

In this article, we continue with the study and development of an important topic in mathematics which are inequalities, particularly inequalities in fractional context. We prove two weighted versions of the generalized Ostrowsky inequality with a weight function $w(t)$, as a consequence of these results we prove conformable and nonconformable fractional versions of this inequality, with the choice of the function $w(t)=(1/(t-a)^{1-\alpha})$ for the conformable case and $w(t)=(1/e^{-\alpha(t-a)})$ for the nonconformable case, these functions have the form $(1/T(t,\alpha))$, where $T(t,\alpha)$ represents the kernel of the fractional integral operator $J^{\alpha}_{T,a}$. $L^p(\varphi,\rangle^{\alpha})$

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest.

Authors' Contributions

The authors contributed equally to this work.

Acknowledgments

The research of Ricardo Abreu-Blaya, José M. Rodríguez and José M. Sigarreta was supported by a grant from Agencia Estatal de Investigación (Grant no. PID2019-106433GB-I00/AEI/10.13039/501100011033), Spain. Partial financial support from ANID Chile: FONDECYT, under Grant no. 1201403, and STIC-AMSUD 22-STIC-09 is gratefully acknowledged by the author Paul Bosch.

References

[1] G. H. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities*, Cambridge University Press, Cambridge, UK, 1934.

[2] E. Locia, A. Morales, J. L. Sánchez, and J. M. Sigarreta, "Epistemological study of mathematical inequalities," *Revista Brasileira de História da Matemática*, vol. 22, no. 43, pp. 68–101, 2021.

[3] P. Bosch, H. J. Carmenate, J. M. Rodríguez, and J. M. Sigarreta, "Generalized inequalities involving fractional operators of the riemann-liouville type," *AIMS Mathematics*, vol. 7, no. 1, pp. 1470–1485, 2021.

[4] J. E. Nápoles Valdés, P. M. Guzman, L. M. Lugo, and A. Kashuri, "The local generalized derivative and mittag-leffler function," *Sigma Journal of Engineering and Natural Sciences*, vol. 38, no. 2, pp. 1007–1017, 2020.

[5] P. Korus, L. M. Lugo, and J. E. Nápoles Valdés, "Integral inequalities in a generalized context," *Studia Scientiarum Mathematicarum Hungarica*, vol. 57, no. 3, pp. 312–320, 2020.

[6] Z. Dahmani, "On Minkowski and Hermite-Hadamard integral inequalities via fractional integration," *Annals of Functional Analysis*, vol. 1, pp. 51–58, 2010.

[7] J. Han, P. O. Mohammed, and H. Zeng, "Generalized fractional integral inequalities of hermite-hadamard-type for a convex function," *Open Mathematics*, vol. 18, no. 1, pp. 794–806, 2020.

[8] K. S. Nisar, F. Qi, G. Rahman, S. Mubeen, and M. Arshad, "Some inequalities involving the extended gamma function and the kummer confluent hypergeometric K-function," *Journal of Inequalities and Applications*, vol. 2018, no. 1, p. 135, 2018.

[9] P. O. Mohammed and F. K. Hamasalh, "New conformable fractional integral inequalities of hermite-hadamard type for convex functions," *Symmetry*, vol. 11, no. 2, p. 263, 2019.

[10] S. Mubeen, S. Habib, and M. N. Naeem, "The minkowski inequality involving generalized k-fractional conformable integral," *Journal of Inequalities and Applications*, vol. 2019, no. 1, p. 81, 2019.

[11] J. E. Nápoles Valdés, P. M. Guzman, and L. M. Lugo, "Some new results on nonconformable fractional calculus," *Advances in Dynamical Systems and Applications*, vol. 13, no. 2, pp. 167–175, 2018.

[12] F. Qi, S. Habib, S. Mubeen, and M. Nawaz Naeem, "Generalized k-fractional conformable integrals and related inequalities," *AIMS Mathematics*, vol. 4, no. 3, pp. 343–358, 2019.

[13] S. Rashid, M. Aslam Noor, K. Inayat Noor, and Y.-M. Chu, "Ostrowski type inequalities in the sense of generalized K-fractional integral operator for exponentially convex functions," *AIMS Math*, vol. 5, no. 3, pp. 2629–2645, 2020.

[14] S. Y. Feng and D. C. Chang, "Boundedness and approximation of the chandrasekhar integral operators in L^Pspaces," *Journal of Nonlinear and Variational Analysis*, vol. 5, pp. 683–707, 2021.

[15] H. Wang, "Certain integral inequalities related to (φ, $>^{\alpha}$)-lipschitzian mappings and generalized h-convexity on fractal sets," *Journal of Nonlinear Functional Analysis*, vol. 2021, Article ID 12, 2021.

[16] S. S. Dragomir and T. M. Rassias, *Ostrowski Type Inequalities and Applications in Numerical Integration*, Springer, New York, NY, USA, 2002.

[17] A. Ostrowski, "Uber die Absolutabweichung einer di erentienbaren funktionen von ihren Integralimittelwert," *Commentarii Mathematici Helvetici*, vol. 10, pp. 226-227, 1938.

[18] S. S. Dragomir and S. Wang, "A new inequality of ostrowski's type in Lp norm," *Indian Journal of Mathematics*, vol. 40, no. 3, pp. 299–304, 1998.

[19] D. Baleanu, J. H. Asad, and A. Jajarmi, "New aspects of the motion of a particle in a circular cavity," *Proceedings of the Romanian Academy Series A*, vol. 19, pp. 143–149, 2018.

[20] D. Baleanu, J. H. Asad, and A. Jajarmi, "The fractional model of spring pendulum: new features within different kernels," *Proceedings of the Romanian Academy Series A*, vol. 19, pp. 447–454, 2010.

[21] D. Baleanu, A. Jajarmi, and J. H. Asad, "Classical and fractional aspects of two coupled pendulums," *Romanian Reports in Physics*, vol. 71, no. 1, p. 103, 2019.

[22] D. Baleanu, S. Sadat Sajjadi, A. Jajarmi, and J. H. Asad, "New features of the fractional euler-lagrange equations for a physical system within non-singular derivative operator," *European Physical Journal—Plus*, vol. 134, no. 4, p. 181, 2019.

[23] R. Khalil, M. Al Horani, A. Yousef, and M. Sababheh, "A new definition of fractional derivative," *Journal of Computational and Applied Mathematics*, vol. 264, pp. 65–70, 2014.

[24] M. ALHorani and R. Khalil, "Total fractional differentials with applications to exact fractional differential equations," *International Journal of Computer Mathematics*, vol. 95, pp. 1444–1452, 2018.

[25] A. Atangana, D. Baleanu, and A. Alsaedi, "New properties of conformable derivative," *Open Mathematics*, vol. 13, no. 1, pp. 889–898, 2015.

[26] A. Atangana and E. F. Doungmo Goufo, "Extension of matched asymptotic method to fractional boundary layers problems," *Mathematical Problems in Engineering*, vol. 2014, Article ID 107535, 1 page, 2014.

[27] A. Atangana and D. Baleanu, "New fractional derivatives with nonlocal and non-singular kernel: theory and application to heat transfer model," *Thermal Science*, vol. 20, no. 2, pp. 763–769, 2016.

[28] R. A. Blaya, R. Á. Ávila, and J. B. Reyes, "Boundary value problems with higher order lipschitz boundary data for polymonogenic functions in fractal domains," *Applied Mathematics and Computation*, vol. 269, pp. 802–808, 2015.

[29] A. Fernandez, M. A. Özarslan, and D. Baleanu, "On fractional calculus with general analytic kernels," *Applied Mathematics and Computation*, vol. 354, pp. 248–265, 2019.

[30] B. Shiri and D. Baleanu, "System of fractional differential algebraic equations with applications," *Chaos, Solitons & Fractals*, vol. 120, pp. 203–212, 2019.

[31] R. Almeida, M. Guzowska, and T. Odzijewicz, "A remark on local fractional calculus and ordinary derivatives," *Open Mathematics*, vol. 14, no. 1, pp. 1122–1124, 2016.

[32] P. M. Guzman, G. Langton, L. M. Lugo, J. Medina, and J. E. Nápoles Valdés, “A new definition of a fractional derivative of local type,” *Journal of Mathematical Analysis*, vol. 9, no. 2, pp. 88–98, 2018.
[33] A. Fleitas, J. F. Gómez-Aguilar, J. E. Nápoles Valdés, J. M. Rodríguez, and J. M. Sigarreta, “Analysis of the local Drude model involving the generalized fractional derivative,” *Optik*, vol. 193, Article ID 163008, 2019.
[34] A. Fleitas, J. E. Nápoles Valdés, J. M. Rodríguez, and J. M. Sigarreta-Almira, “Note on the generalized conformable derivative,” *Revista de la Unión Matemática Argentina*, vol. 62, pp. 443–457, 2021.
[35] P. Bosch, J. F. Gómez-Aguilar, J. M. Rodríguez, and J. M. Sigarreta, “Analysis of dengue fever outbreak by generalized fractional derivative,” *Fractals*, vol. 28, no. 8, Article ID 2040038, 2020.