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ANALYSIS OF DENGUE FEVER OUTBREAK BY GENERALIZED FRACTIONAL DERIVATIVE

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Abstract

In this paper, we use the generalized fractional derivative in order to study the fractional differential equation associated with a fractional Gaussian model. Moreover, we propose new properties of generalized differential and integral operators. As a practical application, we estimate the order of the derivative of the fractional Gaussian models by solving an inverse problem involving real data on the dengue fever outbreak.

Keywords: Fractional Derivatives; Fractional Integrals; Gaussian Fractional Model; Applications.

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1. INTRODUCTION

Fractional calculus was successfully used to model a broad range of phenomena that occur in physics, economy and science. For example, it has been observed that the time or space evolution of many physical processes can be more precisely described when derivatives of fractional order are introduced. In practice, and in many occasions, all that needs to be done is replacing the ordinary derivative in an equation by a derivative of fractional order. Also, the most interesting part is that this is not the result of chance; on the contrary, there is a strong mathematical foundation for it and there are many applications, as several general studies show (cf. Refs. 1–10). For a complementary study on the recent developments in the field of the fractional calculus as well as its applications, see Refs. 11 and 12. Fractional operators keep track of the history of the process being studied; this feature allows modeling the nonlocal and distributed responses that commonly appear in natural and physical phenomena. On the other hand, one has to recognize that these fractional derivatives D^{α} show some drawbacks. To overcome some of these and other difficulties, Khalil *et al.*[13] came up with an interesting idea that generalizes the familiar limit definition of the derivative and allows to introduce successfully a conformable fractional derivative (see Definition 3 below); more recently, a nonconformable fractional derivative is introduced in Ref. 14. In this way, a new direction in fractional calculus was opened, which has shown to be interesting from a theoretical viewpoint and useful in the applications.

According to World Health Organization (WHO), vector-borne diseases account for more than 17% of all infectious diseases, and cause more than 700,000 deaths each year. Among them, more than 3.9 billion people in more than 128 countries are at the risk of dengue. Vector-borne diseases are undoubtedly one of the main health challenges worldwide, given the presence of disease vectors in almost all countries. The efforts of the researchers have focused on the detection and subsequent treatment of infections, however, research and development of technologies that contribute to the prevention of infections have not been sufficient. Understanding the dynamics of vectors (mosquito *Aedes aegypti*) and the population infected with dengue has the potential to generate early warnings for the prevention and containment of the spread of the disease.

Qureshi and Atangana[15] present a comparative study of the dynamics of the dengue fever outbreak under three fractional order differential operators; Liouville–Caputo, Caputo–Fabrizio and Atangana–Baleanu. The true field data for the dengue fever outbreak was used. In this approach, the fractional derivative accomplishes a better fit for the curves of human and mosquito populations but they did not analyze the results associated with the solution of the inverse problems in the local or global approaches.

This paper has been organized as follows: In Sec. 2, we present some fundamental concepts from fractional calculus needed for the convenience of readers. Section 3 is devoted to develop the main results presented in this work. In particular, the lineal–fractional conformable derivative equations are studied using the previous results, obtaining results both for the general solution and for the stability of the solutions. In Sec. 4, a practical application of the Gaussian model is made when solving an inverse problem that involves real data of dengue fever outbreak. Section 5 is devoted to conclusions.

2. PRELIMINARIES

In Ref. 16 is defined the following generalized fractional derivative, which generalizes the conformable fractional derivative in Ref. 13 for the order $0 < \alpha \leq 1$.

Definition 1. Given an interval I, $f : I \to \mathbb{R}$, $\alpha \in \mathbb{R}^+$ and a positive continuous function $T(t, \alpha)$ on I, the *derivative* $G_T^{\alpha} f$ of f of order α at the point $t \in I$ is

$$G_T^{\alpha} f(t) = \lim_{h \to 0} \frac{1}{h^{\lceil \alpha \rceil}} \sum_{k=0}^{\lceil \alpha \rceil} (-1)^k \binom{\lceil \alpha \rceil}{k} \times f(t - khT(t, \alpha)), \tag{1}$$

where $\lceil \cdot \rceil$ represents the upper integer part function.

If $a = \min\{t \in I\}$ (respectively, $b = \max\{t \in I\}$), then $G_T^{\alpha} f(a)$ [respectively, $G_T^{\alpha} f(b)$] is defined with $h \to 0^-$ (respectively, $h \to 0^+$) instead of $h \to 0$ in the limit.

If $T(t, \alpha) = t^{\lceil \alpha \rceil - \alpha}$, then we obtain the following particular case of G_T^{α}, defined in Ref. 13. Note that $T(t, \alpha) = t^{\lceil \alpha \rceil - \alpha} = 1$ for every $\alpha \in \mathbb{N}$.

Definition 2. Let I be an interval $I \subseteq (0,\infty)$, $f : I \to \mathbb{R}$ and $\alpha \in \mathbb{R}^+$. The *conformable derivative* $G^\alpha f$ of f of order α at the point $t \in I$ is defined by

$$G^\alpha f(t) = \lim_{h\to 0} \frac{1}{h^{\lceil\alpha\rceil}} \sum_{k=0}^{\lceil\alpha\rceil} (-1)^k \binom{\lceil\alpha\rceil}{k} \times f(t - kht^{\lceil\alpha\rceil-\alpha}). \tag{2}$$

We know from the classical calculus that if f is defined in a neighborhood of the point t, and there exists $D^n f(t)$, then

$$D^n f(t) = \lim_{h\to 0} \frac{1}{h^n} \sum_{k=0}^{n} (-1)^k \binom{n}{k} f(t - kh).$$

Therefore, if $\alpha = n \in \mathbb{N}$ and f is smooth enough, then Definition 2 coincides with the classical definition of the nth derivative.

In Ref. 13 is defined a conformable derivative in the following way.

Definition 3. Given $f : (0,\infty) \to \mathbb{R}$ and $\alpha \in (0,1]$, the derivative $T_\alpha f$ at the point t is

$$T_\alpha f(t) = \lim_{h\to 0} \frac{f(t) - f(t - ht^{1-\alpha})}{h}. \tag{3}$$

T_α is a particular case of G^α when $\alpha \in (0,1]$ and $T(t,\alpha) = t^{1-\alpha}$. The conformable derivatives defined in Ref. 17 also are particular cases of G^α when $\alpha \in (0,1]$ and $T(t,\alpha) = (t-a)^{1-\alpha}$ and $T(t,\alpha) = (b-t)^{1-\alpha}$. Note that if $\alpha \in (0,1]$ and $T(t,\alpha) = k(t)^{1-\alpha}$, then the general conformable fractional derivative defined in Ref. 18 is derived. Moreover, if $\alpha \in (0,1]$ and $T(t,\alpha) = e^{t^{-\alpha}}$, then the nonconformable fractional derivative defined in Ref. 14 is obtained.

The following results in Refs. 13 and 16 contain some basic properties of the derivative G_T^α.

Theorem 4. *Let I be an interval $I \subseteq \mathbb{R}$, $f : I \to \mathbb{R}$ and $\alpha \in \mathbb{R}^+$. We have the following*:

(1) *If there exists $D^{\lceil\alpha\rceil} f$ at the point $t \in I$, then f is G_T^α-differentiable at t and $G_T^\alpha f(t) = T(t,\alpha)^{\lceil\alpha\rceil} D^{\lceil\alpha\rceil} f(t)$.*
(2) *If $\alpha \in (0,1]$, then f is G_T^α-differentiable at $t \in I$ if and only if f is differentiable at t; furthermore, if f is differentiable at t in the classical sense, then $G_T^\alpha f(t) = T(t,\alpha) f'(t)$.*

Theorem 5. *Let I be an interval $I \subseteq \mathbb{R}$, $f, g : I \to \mathbb{R}$ and $\alpha \in \mathbb{R}^+$. Assume that f, g are G_T^α-differentiable functions at $t \in I$. Then the following statements hold*:

(1) *$af + bg$ is G_T^α-differentiable at t for every $a, b \in \mathbb{R}$, and*

$$G_T^\alpha(af + bg)(t) = a\,G_T^\alpha f(t) + b\,G_T^\alpha g(t).$$

(2) *If $\alpha \in (0,1]$, then fg is G_T^α-differentiable at t and*

$$G_T^\alpha(fg)(t) = f(t) G_T^\alpha g(t) + g(t) G_T^\alpha f(t).$$

(3) *If $\alpha \in (0,1]$ and $g(t) \neq 0$, then f/g is G_T^α-differentiable at t and*

$$G_T^\alpha \left(\frac{f}{g}\right)(t) = \frac{g(t) G_T^\alpha f(t) - f(t) G_T^\alpha g(t)}{g(t)^2}.$$

(4) *$G_T^\alpha(\lambda) = 0$, for every $\lambda \in \mathbb{R}$.*
(5) *$G_T^\alpha(t^p) = \frac{\Gamma(p+1)}{\Gamma(p-\lceil\alpha\rceil+1)} t^{p-\lceil\alpha\rceil} T(t,\alpha)^{\lceil\alpha\rceil}$ for every $p \in \mathbb{R}\backslash\mathbb{Z}^-$.*
(6) *$G_T^\alpha(t^{-n}) = (-1)^{\lceil\alpha\rceil} \frac{\Gamma(n+\lceil\alpha\rceil)}{\Gamma(n)} t^{-n-\lceil\alpha\rceil} T(t,\alpha)^{\lceil\alpha\rceil}$ for every $n \in \mathbb{Z}^+$.*

Theorem 6. *Let $\alpha \in (0,1]$, g be a G_T^α-differentiable function at t and f be a differentiable function at $g(t)$. Then $f \circ g$ is G_T^α-differentiable at t, and $G_T^\alpha(f \circ g)(t) = f'(g(t)) G_T^\alpha g(t)$.*

3. NEW RESULTS

The following result shows an advantage of our definition over some previous definitions of fractional derivatives: a general rule in order to compute iterated derivatives.

Theorem 7. *Let $I \subseteq \mathbb{R}$ be an interval, $t \in I$, $f : I \to \mathbb{R}$, $n \geq 2$ and $\alpha_1, \alpha_2, \ldots, \alpha_n \in (0,1]$. Assume that $T^{(n-k)}(t,\alpha_k)$ exists for $1 \leq k \leq n-1$. Let us define $\overline{T}_k = (T(t,\alpha_k), T'(t,\alpha_k), T''(t,\alpha_k), \ldots, T^{(n-k)}(t,\alpha_k))$ if $1 \leq k < n$, and $\overline{T}_n = T(t,\alpha_n)$. Then $G_T^{\alpha_n} G_T^{\alpha_{n-1}} \cdots G_T^{\alpha_1} f(t)$ exists if and only if $f^{(n)}(t)$ exists, and we have in this case*

$$G_T^{\alpha_n} G_T^{\alpha_{n-1}} \cdots G_T^{\alpha_1} f(t) = Q_n(t) f^{(n)}(t) + P_n(t),$$

where $P_n = p_n(\overline{T}_1, \ldots, \overline{T}_n, f', \ldots, f^{(n-1)})$, p_n is a polynomial on $(n^2+3n-2)/2$ variables which is homogeneous of degree $n+1$, and

$$Q_n(t) = T(t,\alpha_n) T(t,\alpha_{n-1}) \cdots T(t,\alpha_2) T(t,\alpha_1).$$

Proof. We will proceed by induction on n. For $n=2$, Theorem 5 gives $G_T^{\alpha_1} f = T(t,\alpha_1) f'$. If f''

exists, since $T(t,\alpha_1)$ is differentiable, Theorems 4 and 5 give

$$G_T^{\alpha_2} G_T^{\alpha_1} f = T(t,\alpha_2) T(t,\alpha_1) f''(t) + T(t,\alpha_2) T'(t,\alpha_1) f'(t).$$

Note that in this case we have $P_2(t) = T(t,\alpha_2)$ $T'(t,\alpha_1) f'(t)$ and $p_2(x_1,x_2,x_3,x_4) = x_2 x_3 x_4$ is a polynomial of four variables and homogeneous of degree 3.

If $G_T^{\alpha_2} G_T^{\alpha_1} f$ exists, since $\alpha_2 \in (0,1]$, $G_T^{\alpha_1} f = T(t,\alpha_1) f'$ is differentiable by Theorem 4, and so, there exists

$$f^{(2)}(t) = (f'(t))' = \left(\frac{T(t,\alpha_1) f'(t)}{T(t,\alpha_1)} \right)',$$

since $T(t,\alpha_1) > 0$.

Assume now that the induction hypothesis holds for $n-1$ and let us prove it for n.

By induction hypothesis we have

$$G_T^{\alpha_{n-1}} \cdots G_T^{\alpha_1} f(t) = Q_{n-1}(t) f^{(n-1)}(t) + P_{n-1}(t),$$

where

$$P_{n-1} = p_{n-1}(\hat{T}_1, \ldots, \hat{T}_{n-1}, f', \ldots, f^{(n-2)}),$$

where p_{n-1} is a polynomial on $(n^2+n-4)/2$ variables which is homogeneous of degree n,

$$\hat{T}_k = (T(t,\alpha_k), T'(t,\alpha_k), T''(t,\alpha_k), \ldots, T^{(n-1-k)}(t,\alpha_k)),$$

if $1 \le k < n-1$, and $\hat{T}_{n-1} = T(t,\alpha_{n-1})$.

Note that Q_{n-1} and P_{n-1} are differentiable, since there exist the derivatives of $\hat{T}_1, \ldots, \hat{T}_{n-1}, f', \ldots, f^{(n-2)}$ and p_{n-1} is a polynomial. Note also that $Q'_{n-1}(t)$ and $P'_{n-1}(t)$ are homogeneous polynomials of degrees $n-1$ and n, respectively. P'_{n-1} has $(n^2 + 3n-4)/2$ variables $(\hat{T}_1, \ldots, \hat{T}_{n-1}, f', \ldots, f^{(n-2)}, T^{(n-1)}(t,\alpha_1), T^{(n-2)}(t,\alpha_2), \ldots, T'(t,\alpha_{n-1}), f^{(n-1)}) = (\overline{T}_1, \ldots, \overline{T}_n, f', \ldots, f^{(n-1)})$ and the set of variables of Q'_{n-1} is a subset of the variables of P'_{n-1}.

Then, we have

$$G_T^{\alpha_n} G_T^{\alpha_{n-1}} \cdots G_T^{\alpha_1} f(t) = G_T^{\alpha_n}(Q_{n-1} f^{(n-1)} + P_{n-1})(t).$$

Since $\alpha_n \in (0,1]$, then, by Theorems 4 and 5, we have that $G_T^{\alpha_n} G_T^{\alpha_{n-1}} \cdots G_T^{\alpha_1} f(t)$ exists if and only if $(Q_{n-1}(t) f^{(n-1)}(t) + P_{n-1}(t))'$ exists.

If $f^{(n)}$ exists, then $G_T^{\alpha_n} G_T^{\alpha_{n-1}} \cdots G_T^{\alpha_1} f(t)$ exists since Q_{n-1} and P_{n-1} are differentiable. If $G_T^{\alpha_n} G_T^{\alpha_{n-1}} \cdots G_T^{\alpha_1} f(t)$ exists, then $\big(Q_{n-1}(t) f^{(n-1)}(t) + P_{n-1}(t)\big)'$ exists, and since Q_{n-1} and P_{n-1} are differentiable and $Q_{n-1} > 0$, there exists

$$f^{(n)}(t) = (f^{(n-1)}(t))' = \left(\frac{Q_{n-1}(t) f^{(n-1)}(t) + P_{n-1}(t) - P_{n-1}(t)}{Q_{n-1}(t)} \right)'.$$

Hence, $G_T^{\alpha_n} G_T^{\alpha_{n-1}} \cdots G_T^{\alpha_1} f(t)$ exists if and only if $f^{(n)}(t)$ exists; in this case, we have

$$\begin{aligned} G_T^{\alpha_n}(Q_{n-1} f^{(n-1)} + P_{n-1}) &= T(t,\alpha_n) Q_{n-1}(t) f^{(n)}(t) \\ &\quad + T(t,\alpha_n) Q'_{n-1}(t) f^{(n-1)}(t) \\ &\quad + T(t,\alpha_n) P'_{n-1}(t). \end{aligned}$$

Since Q'_{n-1} and P'_{n-1} are homogeneous polynomial of degrees $n-1$ and n, respectively, then

$$P_n(t) = T(t,\alpha_n) Q'_{n-1}(t) f^{(n-1)}(t) + T(t,\alpha_n) P'_{n-1}(t)$$

is a homogeneous polynomial of degree $n+1$ on $(n^2+3n-2)/2$ (the set of variables of P'_{n-1} and $T(t,\alpha_n)$), and this finishes the proof. □

We now provide a more explicit formula for the previous result. Recall that we denote by D the usual derivative.

Theorem 8. *Let I be an interval, $I \subseteq \mathbb{R}$, $t \in I$, $f : I \to \mathbb{R}$, $n \ge 1$ and $\alpha_1, \alpha_2, \ldots, \alpha_n \in (0,1]$. If $f^{(n)}(t)$ and $T^{(n-i)}(t,\alpha_i)$ exist for $1 \le i \le n-1$, then*

$$G_T^{\alpha_n} G_T^{\alpha_{n-1}} \cdots G_T^{\alpha_1} f(t) = T(t,\alpha_n)\, b_{n-1,1}(t),$$

where

$$b_{0,m}(t) = f^{(m)}(t), \quad 1 \le m \le n,$$

$$b_{k,m}(t) = \sum_{j=0}^{m} \binom{m}{j} T^{(j)}(t,\alpha_k) \times b_{k-1,m+1-j}(t), \quad 1 \le k \le n-1, \; 1 \le m \le n-k.$$

Proof. First we show, by induction on k, that $b_{k,m}(t)$ exists for $1 \le k \le n-1$ and $1 \le m \le n-k$.

We have, for $k=1$ and $1 \leq m \leq n-1$,

$$b_{1,m}(t) = \sum_{j=0}^{m} \binom{m}{j} T^{(j)}(t,\alpha_1) b_{0,m+1-j}(t)$$

$$= \sum_{j=0}^{m} \binom{m}{j} T^{(j)}(t,\alpha_1) f^{(m+1-j)}(t).$$

By hypothesis, $T^{(j)}(t,\alpha_1)$ exists for $1 \leq j \leq m \leq n-1$ and since $m+1-j \leq m+1 \leq n$ then $f^{(m+1-j)}(t)$ exists too, and so $b_{1,m}(t)$ exists for $1 \leq m \leq n-1$.

Assume now that the induction hypothesis holds for $k-1 \geq 1$, i.e. $b_{k-1,m}(t)$ exists for $1 \leq m \leq n-k+1$, and let us prove that $b_{k,m}(t)$ exists for $1 \leq m \leq n-k$. We have that

$$b_{k,m}(t) = \sum_{j=0}^{m} \binom{m}{j} T^{(j)}(t,\alpha_k) b_{k-1,m+1-j}(t).$$

By hypothesis, $T^{(j)}(t,\alpha_k)$ exists for $1 \leq j \leq m \leq n-k$ and since $m \leq n-k < n-k+1$ then $m+1-j \leq m+1 \leq n-k+1$ and the induction hypothesis gives that $b_{k-1,m+1-j}(t)$ exists for $0 \leq j \leq m$.

Let us prove the formula $D(b_{k,m}) = b_{k,m+1}$ for $1 \leq m \leq n-k-1$, by induction on $0 \leq k \leq n-1$.

The formula trivially holds for $k=0$ and $1 \leq m \leq n-1$.

Assume that $D(b_{k,m}) = b_{k,m+1}$ for every $0 \leq k$ and $1 \leq m \leq n-k-1$ with $k < n-1$ and let us show that $D(b_{k+1,m}) = b_{k+1,m+1}$ for $1 \leq m \leq n-k-2$:

$$\begin{aligned}
&D(b_{k+1,m}(t)) \\
&\quad = D\left(\sum_{i=0}^{m} \binom{m}{i} T^{(i)}(t,\alpha_{k+1}) b_{k,m+1-i}(t)\right) \\
&\quad = \sum_{i=0}^{m} \binom{m}{i} (T^{(i)}(t,\alpha_{k+1}) D(b_{k,m+1-i}(t)) \\
&\qquad + T^{(i+1)}(t,\alpha_{k+1}) b_{k,m+1-i}(t)) \\
&\quad = \sum_{j=0}^{m} \binom{m}{j} \Bigg(T^{(j)}(t,\alpha_{k+1}) b_{k,m+2-j}(t) \\
&\qquad + \sum_{j=1}^{m+1} \binom{m}{j-1} T^{(j)}(t,\alpha_{k+1})\, b_{k,m+2-j}(t)\Bigg) \\
&\quad = \sum_{j=0}^{m+1} \binom{m+1}{j} T^{(j)}(t,\alpha_{k+1}) b_{k,m+2-j}(t) \\
&\quad = b_{k+1,m+1}(t),
\end{aligned}$$

since $m+1-j \leq m+1 \leq n-k-1$.

Let us prove now the theorem by induction on n. If $n=1$ then Theorem 4 gives $G_T^{\alpha_1} f(t) = T(t,\alpha_1) f'(t) = T(t,\alpha_1) b_{0,1}(t)$, and the formula holds in this case.

Assume now that the induction hypothesis holds for $n-1 \geq 1$ and let us prove it for n. Induction hypothesis and Theorem 4 give

$$\begin{aligned}
&G_T^{\alpha_n} G_T^{\alpha_{n-1}} \cdots G_T^{\alpha_1} f(t) \\
&\quad = G_T^{\alpha_n}(T(t,\alpha_{n-1}) b_{n-2,1}(t)) \\
&\quad = T(t,\alpha_n)[T(t,\alpha_{n-1}) D(b_{n-2,1}(t)) \\
&\qquad + T'(t,\alpha_{n-1}) b_{n-2,1}(t)] \\
&\quad = T(t,\alpha_n)[T(t,\alpha_{n-1}) b_{n-2,2}(t) \\
&\qquad + T'(t,\alpha_{n-1}) b_{n-2,1}(t)] \\
&\quad = T(t,\alpha_n) b_{n-1,1}(t),
\end{aligned}$$

since $1 \leq n-(n-2)-1$, and the formula holds for n. □

As a particular case from the above results, and considering the kernel function $T(t,\alpha) = t^{1-\alpha}$ for $\alpha \in (0,1]$, the following corollary is immediately obtained.

Corollary 9. *Let I be an interval $I \subseteq (0,\infty)$, $f : I \to \mathbb{R}$ and $\alpha_1, \alpha_2, \ldots, \alpha_n \in (0,1]$. If there exists $D^n f(t)$ for some $t \in I$, then*

$$G^{\alpha_n} \cdots G^{\alpha_2} G^{\alpha_1} f(t) = t^{-\alpha_1-\alpha_2-\cdots-\alpha_n} \sum_{k=1}^{n} b_{n,k} t^k f^{(k)}(t),$$

with

$$b_{n,k} = \begin{cases} 0 & \text{for } k=0,\ n \geq 1, \\ 1 & \text{for } k=n,\ n \geq 1, \\ \prod_{j=1}^{n-1}(1-\alpha_1-\cdots-\alpha_{j-1}) & \text{for } k=1,\ n>1, \\ (k-\alpha_1-\cdots-\alpha_{n-1}) & \text{for } k<n. \\ \quad \times b_{n-1,k} + b_{n-1,k-1} & \end{cases}$$

Let I be an interval, $a, t \in I$ and $\alpha \in \mathbb{R}$. Following Ref. 13, the integral operator $J_{T,a}^{\alpha}$ is defined for every locally integrable function f on I as

$$J_{T,a}^{\alpha}(f)(t) = \int_a^t \frac{f(s)}{T(s,\alpha)} ds.$$

Proposition 10. *Let I be an interval, $a \in I$, $0 < \alpha \leq 1$ and f a differentiable function on I with f'*

locally integrable on I. Then, we have for all $t \in I$ that

$$J_{T,a}^{\alpha}(G_T^{\alpha}(f))(t) = f(t) - f(a).$$

In Ref. 13, it is defined the integral operator $J_{T,a}^{\alpha}$ for the choice of T given by $T(t,\alpha) = t^{1-\alpha}$, and Theorem 3.1 in Ref. 13 shows that

$$G^{\alpha} J_{t^{1-\alpha},a}^{\alpha}(f)(t) = f(t)$$

for every continuous function f on I, $a, t \in I$ and $\alpha \in (0,1]$. Hence, Proposition 11 extends to any T this important equality.

Proposition 11. *Let I be an interval, $a \in I$ and $\alpha \in (0,1]$. Then*

$$G_T^{\alpha}(J_{T,a}^{\alpha}(f))(t) = f(t)$$

for every continuous function f on I and $a, t \in I$.

Theorem 12 in the following contains some elementary facts on the integral operator $J_{T,a}^{\alpha}$.

Theorem 12. *Let I be an interval, $a, b \in I$ and $\alpha \in \mathbb{R}$. Let f, g be locally integrable functions on I, and $k_1, k_2 \in \mathbb{R}$. We have the following*:

(1) $J_{T,a}^{\alpha}(k_1 f + k_2 g)(t) = k_1 J_{T,a}^{\alpha} f(t) + k_2 J_{T,a}^{\alpha} g(t)$;
(2) *if $f \geq g$, then $J_{T,a}^{\alpha} f(t) \geq J_{T,a}^{\alpha} g(t)$ for every $t \in I$ with $t \geq a$;*
(3) $|J_{T,a}^{\alpha} f(t)| \leq J_{T,a}^{\alpha}|f|(t)$ *for every $t \in I$ with $t \geq a$;*
(4) $\int_a^b \frac{f(s)}{T(s,\alpha)} ds = J_{T,a}^{\alpha} f(t) - J_{T,b}^{\alpha} f(t) = J_{T,a}^{\alpha} f(t)(b)$ *for every $t \in I$.*

Given $a, b \in I$ $(b > a)$, let us denote by $F_{a,b}$ the usual inner product in $L^2[a,b]$,

$$F_{a,b}(f,g) = \int_a^b f(t)g(t)dt.$$

Proposition 13. *Let I be an interval, $a, b \in I$ with $a < b$ and $\alpha \in \mathbb{R}$. The adjoint of $J_{T,a}^{\alpha}$ in $L^2[a,b]$ with respect to the inner product $F_{a,b}$ is the operator*

$$A_{T,a,b}^{\alpha}(f)(t) = \frac{1}{T(t,\alpha)} \int_t^b f(s)ds.$$

Proof. We obtain, by using integration by parts,

$$\begin{aligned}
F_{a,b}(J_{T,a}^{\alpha}(f), g) &= \int_a^b g(t) J_{T,a}^{\alpha}(f)(t)dt \\
&= \left[-J_{T,a}^{\alpha}(f)(t) \int_t^b g(s)ds \right]_{t=a}^{t=b} \\
&\quad + \int_a^b \frac{f(t)}{T(t,\alpha)} \int_t^b g(s)dsdt \\
&= \int_a^b \frac{f(t)}{T(t,\alpha)} \int_t^b g(s)dsdt \\
&= \int_a^b f(t) A_{T,a,b}^{\alpha}(g)(t)dt \\
&= F_{a,b}(f, A_{T,a,b}^{\alpha}(g)). \qquad \square
\end{aligned}$$

Let I be an interval in $\mathbb{R}$ and Ω an open set in $\mathbb{R}^n$. The function $H : I \times \Omega \to \mathbb{R}^N$ is *uniformly Lipschitz* with respect to the second variable x if there is a constant L with $|H(t,x) - H(t,y)| \leq L|x - y|$ for every $t \in I$ and $x, y \in \Omega$. The function H is in (C, Lip) on $I \times \Omega$ if it is a continuous function on $I \times \Omega$ and it is uniformly Lipschitz continuous with respect to the second variable on this set.

In Ref. 19 the following result was proved.

Lemma 14. *Let $\alpha_1, \alpha_2, \ldots, \alpha_n \in (0,1]$, $x = (x_1, \ldots, x_n)$, I an interval in $\mathbb{R}$, Ω an open set in $\mathbb{R}^n$, $t_0 \in I$ and $x_0 \in \Omega$. Let $F = (F_1, \ldots, F_n) : I \times \Omega \to \mathbb{R}^n$ be in (C, Lip) on some open neighborhood of the point (t_0, x_0), and consider the initial value problem*

$$\begin{aligned}
G_T^{\alpha_j} x_j &= F_j(t,x), \quad 1 \leq j \leq n, \\
x(t_0) &= x_0.
\end{aligned} \tag{4}$$

Then there exists $h > 0$ such that (4) has a unique solution on $[t_0 - h, t_0 + h] \cap I$.

Furthermore, if $\Omega = \mathbb{R}^n$ and F is in (C, Lip) on $J \times \mathbb{R}^n$ for each compact interval $J \subseteq I$, then (4) has a unique solution on I.

Let $\alpha \in (0,1]$, $a \in \mathbb{R}$, $t_0 \geq a$, $x_0 \in \mathbb{R}^n$ and $F : [a,\infty) \times \mathbb{R}^n \to \mathbb{R}^n$ be in (C, Lip) on $[a,\infty) \times \mathbb{R}^n$. Consider

$$G_T^{\alpha} x(t) = F(t,x), \quad x(t_0) = x_0. \tag{5}$$

Lemma 14 guarantees that Eq. (5) has a unique solution on $[a,\infty)$.

Propositions 15 and 16 give the following result.

Proposition 15. *Let $\alpha \in (0,1]$, $a \in \mathbb{R}$, $t_0 \geq a$, $x_0 \in \mathbb{R}^n$ and $F : [a,\infty) \times \mathbb{R}^n \to \mathbb{R}^n$ be in (C, Lip) on $[a,\infty) \times \mathbb{R}^n$. Then the problem (5) is equivalent to*

$$x(t) = x_0 + J_{T,t_0}^{\alpha} F(s, x(s))(t). \tag{6}$$

The study of boundedness of solutions of a differential equation, either fractional or not, plays an important role in qualitative theory. In addition, the qualitative behavior of solutions is very important in applied research. Using the previous propositions one can check that.

Let I be an interval with $[t_0,\infty) \subseteq I$ and $\alpha \in (0,1]$. Let p be a continuous function on $[t_0,\infty)$. Then the initial value problem

$$G_T^\alpha y + p(t)y = 0, \quad y(t_0) = y_0,$$

has a unique solution

$$y(t) = y_0 e^{-J^\alpha_{T,t_0}(p)(t)}. \tag{7}$$

Proposition 16. *Let I be an interval with $[t_0,\infty) \subseteq I$ and $\alpha \in (0,1]$. Let p be a continuous function on $[t_0,\infty)$, and consider the linear equation*

$$G_T^\alpha y + p(t)y = 0. \tag{8}$$

We have the following:

(1) *If $\liminf_{t\to\infty} J^\alpha_{T,t_0}(p)(t) > -\infty$, then the trivial solution $y(t) \equiv 0$ is stable.*
(2) *If $\lim_{t\to\infty} J^\alpha_{T,t_0}(p)(t) = \infty$, then $y(t) \equiv 0$ is asymptotically stable.*

Proof. The initial value problem (8) with $y(t_0) = y_0$ has a unique solution

$$y(t) = y_0 e^{-J^\alpha_{T,t_0}(p)(t)}. \tag{9}$$

Since $J^\alpha_{T,t_0}(p)$ is a continuous function, if $\liminf_{t\to\infty} J^\alpha_{T,t_0}(p)(t) > -\infty$, then there exists a constant M with $J^\alpha_{T,t_0}(p)(t) \geq M$ for every $t \in [t_0,\infty)$. Hence, $|y(t)| \leq |y_0|e^{-M}$ for every $t \in [t_0,\infty)$. Therefore, for any $\varepsilon > 0$, if we choose y_0 with $|y_0| < \varepsilon e^M$, then $|y(t)| < \varepsilon$ for every $t \in [t_0,\infty)$, and so, the trivial solution is stable.

If $\lim_{t\to\infty} J^\alpha_{T,t_0}(p)(t) = \infty$, then (9) gives $\lim_{t\to\infty} y(t) = 0$, and so, $y(t) \equiv 0$ is asymptotically stable. □

4. THE GAUSSIAN MODEL INVOLVING THE GENERALIZED FRACTIONAL DERIVATIVE

This section is framed in the context of the fractional Gaussian model presented in the following, with $b, c \in \mathbb{R}^+$ and $\alpha \in (0,1]$:

$$G^\alpha f(t) + \frac{t-b}{c^2} f(t) = 0. \tag{10}$$

A broad range of dynamic systems arising in physical problems are depicted by this model.

For a more complete study of the Gaussian model, we will consider some particular cases of interest, in the kernel of definition of the derivative G. Thus, we can consider (among others) the following cases:

Case (a): $T(t,\alpha) = e^{(1-\alpha)t}$. The solution of the system (10) is

$$f(t) = a e^{\frac{c^{-2}}{\alpha-1}\left(\frac{-e^{b(\alpha-1)}}{\alpha-1} + e^{t(\alpha-1)}\left(b + \frac{1}{\alpha-1} - t\right)\right)}. \tag{11}$$

Case (b): $T(t,\alpha) = t^{1-\alpha}$. The solution of the system (10) is

$$f(t) = a \cdot e^{\left(\frac{t^\alpha[(\alpha+1)b - \alpha t] - b^{\alpha+1}}{c^2\alpha(\alpha+1)}\right)}. \tag{12}$$

Note that if $\alpha \to 1$ in (12) and (11), then the solution

$$f(t) = a \cdot \exp\left(\frac{-(t-b)^2}{2c^2}\right) \tag{13}$$

of the ordinary differential equation $f'(t) + \frac{t-b}{c^2} f(t) = 0$ is obtained.

The fractional Gaussian model is studied by using Caputo fractional derivative, through its numerical relation with Grünwald–Letnikov fractional derivative (see Ref. 20).

$$ {}^{\mathrm{GL}}D_a^\alpha f(t_m) \approx h^{-\alpha} \sum_{k=0}^{m} (-1)^k \binom{\alpha}{k} f(t_m - kh),$$
$$t_m = mh, \quad m = 0,1,2,\ldots, \tag{14}$$

$$f(t_m) = h^\alpha \frac{b - t_{m-1}}{c^2} f(t_{m-1}) + \sum_{k=1}^{m} (-1)^{k+1} \binom{\alpha}{k} f(t_{m-1} - kh). \tag{15}$$

Figure 1 shows the curves associated with a direct-type problem for a Gaussian model, with the following values of the parameters: $a = 1$, $b = 1$, $c = 1$ and $\alpha = 0.85$, for the approaches namely ordinary derivative (*black*), fractional conformable derivative according to $T(t,\alpha) = e^{(1-\alpha)t}$ (*red*), Khalil *et al.* (*blue*) and Caputo fractional derivative (*green*).

An observation equation can be related to the model:

$$y_i = g(f(t_i)) + \varepsilon_i, \quad i = 1,\ldots,n, \tag{16}$$

where each y_i corresponds to the ith observed value under uncertainty from a solution of (10)

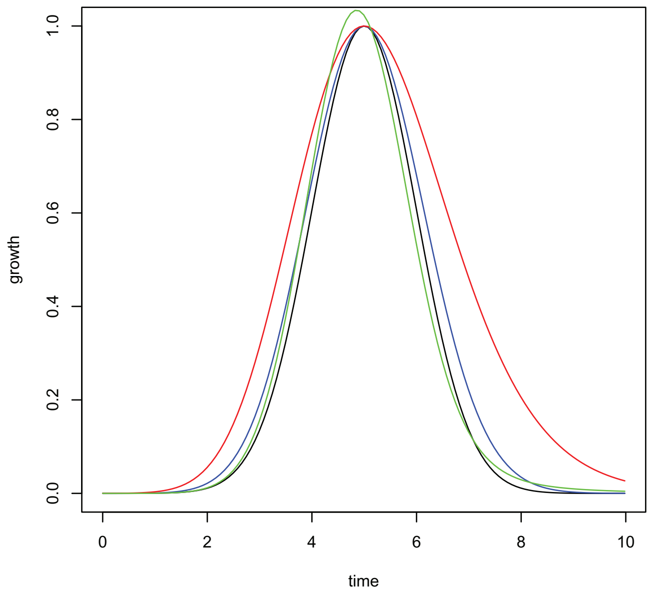

Fig. 1 Gaussian model's curves.

at the discrete time $t_i \in [0;T], i = 1, 2, \ldots, n$; g is the observation function; and ε_i are measurement errors, which are considered as independent and identically distributed (i.i.d.) random variables from a normal distribution, with mean zero and constant variance σ^2, denoted by $\varepsilon_i \sim \mathcal{N}(0, \sigma^2)$.

According to Bayesian statistical inversion theory, and provided that all information available has been assimilated into the model, the solution to an inverse problem is given by the subsequent distribution of the quantity of interest. For the model defined in (10), the parameter of interest is $\theta = (a, b, c, \alpha)$. Next, some prior distributions are proposed,

$$a \sim \mathcal{G}(r_a, \lambda_a), \tag{17}$$

$$b \sim \mathcal{G}(r_b, \lambda_b), \tag{18}$$

$$c \sim \mathcal{G}(r_c, \lambda_c), \tag{19}$$

$$\alpha \sim \mathcal{U}(0, 1), \tag{20}$$

where $\mathcal{G}(r, \lambda)$ denotes the Gamma distribution with shape parameter r and rate parameter λ, $\mathcal{U}$ is the continuous uniform distribution on the interval $(0, 1)$. The parameters introduced in the prior distributions are called hyperparameters. As mentioned before, the prior distributions have been defined in order to reflect the information already known about the possible values of the parameters of interest. In the model (10), a, b and c are positive and $\alpha \in (0, 1]$.

Taking for granted the prior independence of the parameters, the joint prior distribution can be written as

$$p(\theta \,|\, \text{hyperparameters}) = p(a \,|\, r_a, \lambda_a) p(b \,|\, r_b, \lambda_b) \times p(c \,|\, r_c, \lambda_c) p(\alpha), \tag{21}$$

where $p(a \,|\, r_a, \lambda_a)$, $p(b \,|\, r_b, \lambda_b)$, $p(c \,|\, r_c, \lambda_c)$ are $p(\alpha)$ are defined in (17)–(20), respectively. Let $y' = (y_1, y_2, \ldots, y_n)$ denote the i.i.d. observed data at times $(t_1, t_2, \ldots, t_n)$ from the model defined by (10) and (16), the likelihood function is given by

$$L(y \,|\, \theta) = \prod_{i=1}^{n} f_Y(y_i) = \frac{1}{(\sigma\sqrt{2\pi})^n} \exp\left\{ -\frac{1}{2\sigma^2} \sum_{i=1}^{n} \times (y_i - g(f(t_i)))^2 \right\}, \tag{22}$$

where $f(t_i)$, $i = 1, \ldots, n$, is a solution of (10).

Then, applying Bayes' theorem, the posterior distribution of the parameters of interest is given by

$$p(\theta \,|\, y) = \frac{L(y \,|\, \theta) p(\theta)}{\int_{\Theta} L(y \,|\, \theta) p(\theta) d\theta}, \tag{23}$$

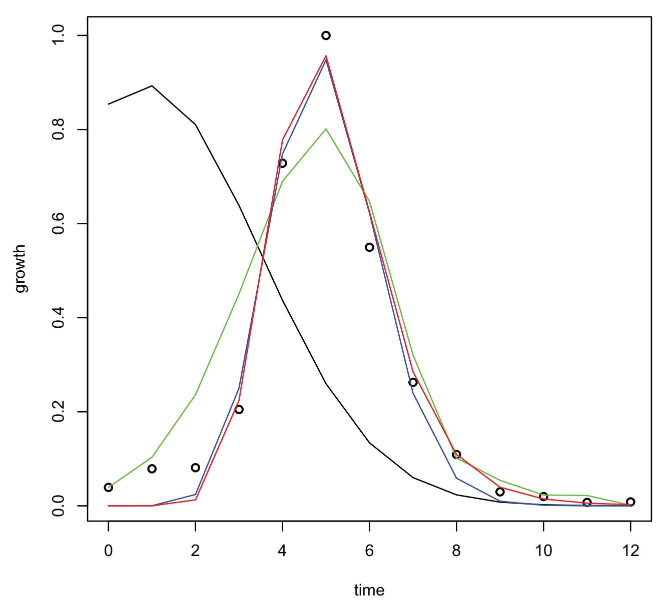

Fig. 2 Data and estimates of dengue.

where $\ominus$ denotes the parameter space of θ. Obviously,

$$
\begin{aligned}
p(\theta \mid y) &\propto L(y \mid \theta) p(\theta) \\
&= \frac{1}{(\sigma\sqrt{2\pi})^n} \cdot \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i - g(f(t_i)))^2\right\} \\
&\quad \cdot \left\{a^{r_a-1}\exp(-a/\lambda_a)\frac{1}{\Gamma(r_a){\lambda_a}^{r_a}}\right\} \\
&\quad \cdot \left\{b^{r_b-1}\exp(-b/\lambda_b)\frac{1}{\Gamma(r_b){\lambda_b}^{r_b}}\right\} \\
&\quad \cdot \left\{c^{r_c-1}\exp(-c/\lambda_c)\frac{1}{\Gamma(r_c){\lambda_c}^{r_c}}\right\}.
\end{aligned} \tag{24}
$$

Adopting a loss quadratic function, the Bayesian point estimation is the posterior mean of $\hat{\theta}_B$, which is given by

$$\hat{\theta}_B = E(\theta \mid \mathbf{y}). \tag{25}$$

Markov chain Monte Carlo (MCMC) simulations are employed to work with (24). Among the most

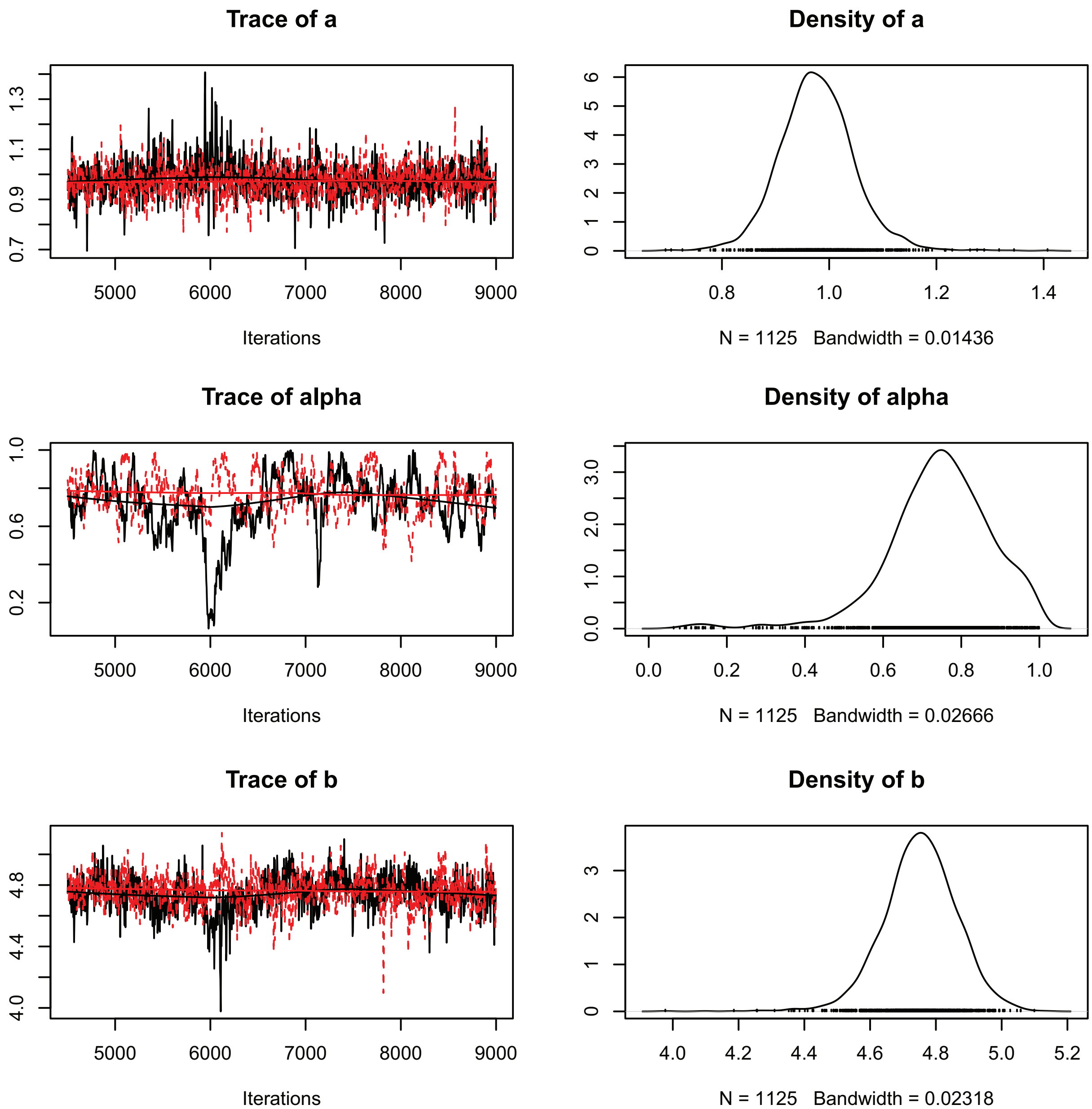

Fig. 3 Trace and estimated posterior densities of a, α and b.

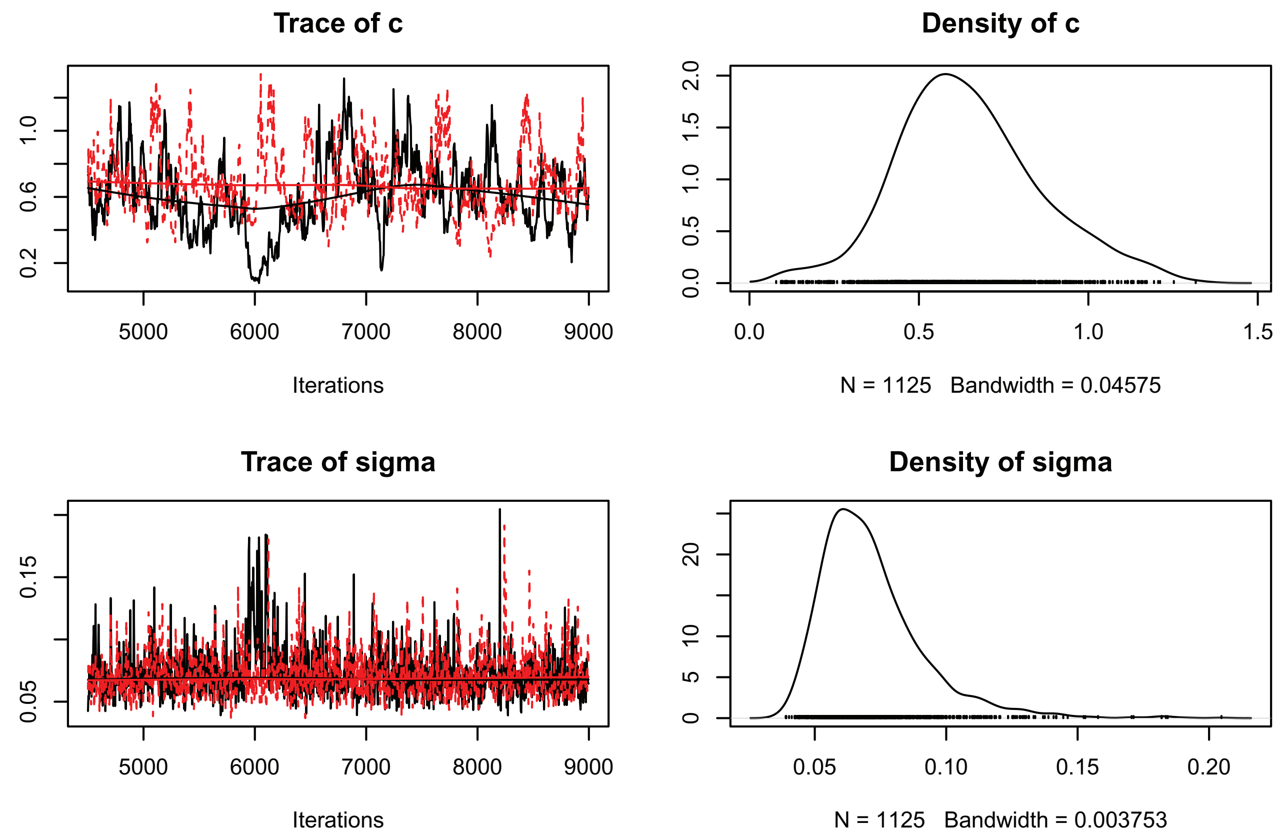

Fig. 4 Trace and estimated posterior densities of c and σ.

popular MCMC techniques are the Gibbs sampling[21] and the Metropolis–Hastings algorithm.[22,23] WinBUGS, JAGS, Stan and t-walk are some computer programs that implement the MCMC algorithms.

Now we show two applications to real problems of the Gaussian model by different approaches namely Khalil *et al.* (*blue*), conformable derivative with $T(t, \alpha) = e^{(1-\alpha)t}$ (*red*), Caputo (*green*) and ordinary (*black*).

For this study, we used data on the dengue fever from the work by Quereshi and Atangana[15] and the kernel function used to define the conformable derivative was $T(t, \alpha) = e^{(1-\alpha)t}$. Figure 2 shows the adjustments corresponding to the observations (black points) associated with dengue fever, according to the different fractional approaches.

The errors in the adjustments according to the different approaches are the following: Khalil *et al.* (0.02534277), conformable derivative with $T(t, \alpha) = e^{(1-\alpha)t}$ (0.02194562), Caputo (0.14031743) and ordinary (0.03062900).

Figures 3 and 4 (dengue fever) show the trace and estimated posterior distributions of the parameters of interest for G_T^α with $T(t, \alpha) = e^{(1-\alpha)t}$.

The estimates of the parameters α, a, b, c and τ related to dengue fever data, according to the different approaches, are as follows:

Fractional Derivative	α	a	b	c	τ
Ordinary	—	0.95701376	4.89718951	1.18583780	0.07703328
G_T^α with $T(t, \alpha) = e^{(1-\alpha)t}$	0.75981801	0.97828067	4.75902783	0.66649858	0.07056143
Khalil *et al.*	0.42466034	0.96194346	4.82362832	0.77065147	0.07170003
Caputo	0.6969704	0.6969704	4.9656985	1.6013685	0.1292894

5. CONCLUSIONS

Fractional calculus is successfully used to model a broad range of problems, since the evolution of many physical processes can be more precisely described with fractional derivatives. We study some general properties of fractional Gaussian model with the conformable fractional derivative. In particular, we used a generalized conformable fractional derivative $[G_T^\alpha$ with $T(t,\alpha) = e^{(1-\alpha)t}]$ in order to study a Gaussian model. Taking into account an experimental dataset, we solve an inverse problem to estimate the order of the involved fractional derivative. From the above results, it is inferred that the conformable approach proposed by us minimizes the error in the adjustments in relation to the other models studied.

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