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Precautionary saving in mean-variance models and different sources of risk[☆]

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ABSTRACT

We study the effects of first- and second-order risk increases on precautionary saving in a mean-variance model. In doing so, we reduce the gap between the theory of saving, which mainly stems from the expected utility model, and empirical estimations of the theory that are based on different measures of dispersion; these are atheoretical concepts that do not arise from optimal agent behavior. We then analyze what effects different risk sources have on saving and show that our results, derived in the mean-variance space, can easily be translated to conditions in the expected utility space. We argue that our contribution establishes a more solid ground for analyzing policies in highly risky environments, such as the COVID-19 pandemic.

1. Introduction

In the past few months, economic uncertainty has risen to levels rarely seen in the last decades. The global pandemic challenges our perspective on the future and has produced an economic collapse in the short term that is aggravated by uncertainty regarding its length and depth. Both firms and consumers have been affected by this increasing uncertainty and have consequently reduced investment, spending, and consumption. Rising unemployment rates have imposed dramatic difficulties for governments around the world regarding how to balance health protection measures with economic recovery; financially stressed firms with drastically reduced cash flows must continue operating in order to avoid exacerbating the economic damage caused by the coronavirus. As a result, uncertainty is compounded by more uncertainty in what seems to be an economic negative feedback loop.

Berlingieri et al. (2017) report that many OECD countries have seen a significant increase in wage dispersion in recent years due to globalization and digitization, which have also increased the uncertainty faced by consumers. Since most consumers obtain their income from labor, savings constitute a useful cushion for weathering negative idiosyncratic shocks to income and potential economic slumps. It is within this context of increasing uncertainty that precautionary saving plays a central role in the economy.

Ordinarily, saving serves as a way to smooth intertemporal consumption. However, in the context of uncertainty, a second reason for saving–precautionary saving–arises. For example, when labor income becomes unpredictable and formal insurance is not available, risk is inescapable for consumers. Given risk-averse consumers, uncertainty about future income is perceived as bad, and this reduces their well-being. Consumers will therefore want to combine labor income that has become random with a good, leading to a level of wealth (and well-being) that is higher while consumers face uncertainty. "Savings" is the control variable that consumers can use to redistribute present wealth to the future, which in turn reduces the pain associated with uncertainty and thus increases their well-being (Eeckhoudt and Schlesinger, 2006; Eeckhoudt et al., 2009).

The idea of additional saving under risk goes back to the writings of Marshall (1920) and Fisher (1930). However, the concept of precautionary saving as we understand it today was first formalized by Leland (1968). For small risks, Leland derived the conditions under which preferences for a positive precautionary effect arise in the case of labor income risk. He called this additional saving the "precautionary

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demand for saving", giving rise to the concept of precautionary saving or the precautionary effect. In the same period, Sandmo (1970) defined a particular instance of a mean-preserving increase in risk and applied it to labor income risk and interest rate risk.[1] Hahn (1970) examined a shock to wealth and was the first, to the best of our knowledge, to propose that an increase in the demand for precautionary saving is connected to the convexity of the marginal utility function $u'''(>0)$. In a later work, Kimball (1990) called this condition "prudence". Kimball's seminal contribution provided the foundation for the modern study of precautionary saving, which stimulated the development of new ideas and resources in risk and uncertainty theory. Baiardi et al. (2020) reviews how the theoretical literature on precautionary saving has developed in recent years. They focus on studies that adapt the classic framework to the problem of intertemporal saving using the expected utility approach, and also provide a brief summary of studies that adopt an approach of non-expected utility, such as ambiguity models.

At the theoretical level, researchers tend to agree that precautionary saving is determined by an individual's preferences for risk, type of risk shock, and risk source. At the empirical level, however, results are not conclusive. Lugilde et al. (2019) review the empirical precautionary saving literature and find that there is no consensus on the most appropriate uncertainty measures for quantifying either the importance of precautionary saving or the intensity of the precautionary effect. Some of these studies use the "precautionary premium" as a measure of uncertainty to quantify the importance of precautionary saving. This concept was introduced by Kimball (1990), who defined it as the certain income reduction that has the same effect on an individual's welfare as adding risk to the expected marginal utility. He specified the precautionary premium using labor income risk, which is an additive and linear risk. Because the precautionary premium is interpreted in the context of a decision model, it is seen as a theoretical measure of uncertainty. However, most empirical work has used variance (or standard deviation) as an uncertainty measure, and since this statistical moment does not arise from a decision model, variance is considered to be an *atheoretical* measure of uncertainty. Consequently, there is a gap between theory and empirics in the study of saving.

In this paper, therefore, we aim to provide a theoretical foundation for the use of variance as a measure of uncertainty in the analysis of precautionary saving. Unlike the precautionary premium, variance is compatible with different types of risk beyond the additive case, and therefore analyzing precautionary saving in the mean-variance (or mean−standard deviation) model, usually denoted by the (μ,σ)-model, reduces the gap between theory and empirical work in the study of precautionary saving.

It is frequently thought that (μ,σ)-analysis is compatible with the expected utility criterion only if the utility function is quadratic or if random variables are normally distributed. However, the theoretical literature on (μ,σ)-preferences emphasizes that these conditions can be rendered more flexible. The pioneering advances of Sinn (1983) and Meyer (1987) determined what conditions are necessary for the expected utility framework—based on the von Neumann-Morgenstern utility functions—and (μ,σ) analysis to be equivalent. These studies provided the theoretical foundations for taking specific concepts originating in the expected utility space and translating them into the (μ,σ) space. The properties were derived for utility functions with only one argument ($u(y)$) and for risks that satisfy the location and scale condition.[2] Thus, (μ,σ)-analysis is more restrictive than the expected utility approach. The latter allows us to analyze precautionary saving based on utility functions using two arguments ($u(y,z)$), nonlinear risks and higher-order stochastic changes.[3] However, the virtue of the (μ,σ)-approach is its simplicity and ease of interpretation for certain phenomena that occur in economics, one of which—as we will see in this paper—is precautionary saving.

This study aims to analyze the effects of first- and second-order increases in risk on saving within the (μ,σ)-space. To this end, we employ a simple two-period model with a separable utility function[4] and linear risks.[5] Both assumptions are required for (μ,σ)-analysis to be a perfect substitute for the method of expected utility (Meyer, 1987). The conditions we provide on (μ,σ)-preferences are necessary and sufficient to guarantee that a first- and second-order increase in risk will produce an increase (decrease) in precautionary saving. In our results, we make use of the elasticity risk-aversion concept of Battermann et al. (2002) in the (μ,σ)-space in order to capture the relationship between increases in risk and the effect on saving. We show that the elasticity of risk aversion reaches a threshold in relation to the mean that is equivalent to a general risk-aversion index, while the elasticity of risk aversion reaches a threshold in relation to the standard deviation that is equivalent to a general prudence index.[6] To test the robustness of our results, we analyze the effects of first- and second-order risk on saving in the expected utility space under the standard definitions of first-order stochastic dominance and a mean-preserving spread (Ekern, 1980). To perform this analysis, we follow Diamond and Stiglitz (1974). We then apply the results to different sources of risk, employing a framework similar to that of Gunning (2010) for uncertain future income.

Our results underline the three-point relevance of studying the effects of uncertainty on saving. First, we bridge the gap between theoretical and empirical applications of precautionary saving by mapping the theory of precautionary saving from the expected utility world into the (μ,σ)-framework. Second, our findings yield better understanding of an agent's economic behavior under risk. Given the current uncertainty shock of the COVID-19 pandemic to the global economy, this issue is critical for both the private sector and policymakers responsible for reactivating severely damaged economic machinery. Finally, we connect our results to economics decision problems previously analyzed in the (μ,σ)-space as a way to highlight both the usefulness of our general model in different applications and the broad potential of the traditional mean-variance approach in economics.

The rest of the paper is organized as follows. In Section 2, we provide a short literature review of recent application of the (μ,σ)-model. In Section 3, we develop a two-period intertemporal saving model in the (μ,σ)-space. In Section 4, we analyze the consequences of an increase in first-order risk and relate the results to the elasticity of risk aversion in terms of the mean (μ). We derive two equivalent general risk-aversion

[1] Rothschild and Stiglitz (1971) also examined the consequences of interest rate risk on the precautionary demand for saving.

[2] The location-scale condition requires that all random variables in the choice set be linearly related to one another. Thus, the (μ,σ)-approach perfectly substitutes the expected utility approach. This property is found in several important decision-making problems in economics: portfolio selection (Eichner and Wagener, 2011; Ormiston and Schlee, 2001); competitive firm under price uncertainty (Meyer, 1987; Hawawini, 1978); insurance demand (Bonilla and Ruiz, 2014; Eichner and Wagener, 2004a, 2004b, 2014; Battermann et al., 2002); banking (Broll et al., 2015); international trade (Broll and Mukherjee, 2017); and economic theory (Eichner and Wagener, 2003, 2004a, 2004b, 2009, 2012; Eichner, 2008; Lajeri and Nielsen, 2000; Wagener, 2002, 2003).

[3] See, for instance, Eeckhoudt et al. (2009); Chiu et al. (2012); Jouini et al. (2013); Crainich et al. (2016).

[4] Leland (1968); Sandmo (1970); and Dardanoni (1988) consider nonseparable utility when analyzing precautionary saving. However, it is customary in intertemporal economics to assume time-separability of the utility function (Baiardi et al., 2020).

[5] This assumption allows location and scale conditions to be satisfied.

[6] Both indices are hybrid, because they have elements of the expected utility and (μ,σ) methods.

indices and apply them to different types of risk. In Section 5, we analyze the consequences of second-order risk and relate the results to the elasticity of risk aversion with respect to the standard deviation (σ). We derive two equivalent general prudence indices and apply them to different types of risk. In Section 6, we discuss the results of our paper as they relate to some decision problems in economics analyzed under the (μ,σ)-approach. Section 7 concludes.

2. Recent literature on applications of the (μ,σ) model

In recent years, there has been a new stream of literature that takes advantage of the Sinn (1983) and Meyer (1987) methodology to translate problems from the expected utility space into the (μ,σ)-paradigm. These new applications have given birth to new concepts like the elasticity of risk aversion. This was established by Battermann et al. (2002) to capture the relationship between a second-order risk increase and insurance demand changes as a function of elasticity. Those authors argued that if the elasticity of risk aversion is greater (less) than unity, then an increase in risk will decrease (increase) the demand for insurance. Eichner and Wagener (2004a, 2004b) connected the results in Battermann et al. (2002) and the index of partial prudence defined by Choi et al. (2001). Bonilla and Ruiz (2014) and Eichner and Wagener (2014) studied the demand for insurance when the decision maker is confronted with a first-order increase in risk. Eichner and Wagener (2014) showed that, if the elasticity of risk aversion exceeds −1, the demand for insurance will decrease with a decrease in the expected size of insurable losses or an increase in the insurance premium. The risk-aversion elasticity is related to a partial relative risk aversion index.

In a different application of the (μ,σ)-model, Broll et al. (2006) characterized the connection between export production and exchange rate uncertainty. They showed that a firm's exports will increase (decrease) if and only if the standard deviation-related elasticity of risk aversion is greater than (less than) unity. Additionally, they showed that an increase in exchange rate expected value will increase a firm's exports if the mean-related elasticity of risk aversion is less than or equal to unity.

More recently, Eichner and Wagener (2012) studied the tempering effect of background risk on decisions under risk, connecting their results with common notions developed in the expected utility space such as risk vulnerability, properness and standardness. Huang and Yang (2020) also studied the inclusion of background risk in the (μ,σ)-model and provided a new characterization of the optimal portfolio frontier for this case.

Broll et al. (2015) analyzed a bank's optimal investment amount when facing changes in the return risk of financial assets. They found that investment will increase when risk (the standard deviation) increases if and only if the standard deviation-related elasticity of risk aversion is less than −1, while investment will increase when the expected value of returns increases if and only if the mean-related elasticity of risk aversion is less than 1.

Broll and Mukherjee (2017) examined a firm's optimal production and trade decisions when it operated in both its domestic market and a foreign market while facing exchange rate uncertainty. The results of their paper showed that the firm reduces its exports when the standard deviation increases if and only if the standard deviation-related elasticity of risk aversion is greater than −1. Given an increase in the mean, exports increase if and only if the mean-related elasticity of risk aversion is less than 1.

Finally, Bi et al. (2018) use a behavioral mean-variance portfolio selection problem in continuous time where cash flow is distorted by a probability distortion function, posing a problem for classical optimization methods. However, by using the decision variable "quantile function of terminal cash flow", they are able to estimate an efficient frontier and exploit such results. Approximations of mean-variance analysis under continuous-time models are also currently being developed (Trybula and Zawisza, 2019; Strub and Li, 2020).

3. Risk and saving in a two-period and two-moment model

We employ a two-period model of consumption with a certain income w in the first period and uncertain income in the second period. The consumer operates under a time-separable utility function $v(c_1) + u(c_2(\theta,s))$, where $c_1 = w - s$ is consumption in the first period and $c_2(\theta,s)$ is consumption in the second period, with the assumption that it is increasing and concave in s.[7] Thus, the utility function is $v(\cdot)$ in the first period and $u(\cdot)$ in the second period. Furthermore, θ is a random variable with mean μ_θ and variance σ_θ, indexing the state of nature and representing the risks that affect future income, and s denotes saving, representing the choice or control variable in the two-period consumption model.

The consumer pursues the optimal savings s^* that will maximize their expected utility of intertemporal consumption. Sinn (1983) and Meyer (1987) have shown that, under the location-scale condition, the expected utility decision problem can be transformed into the (μ,σ)-framework. However, in the intertemporal consumption problem, it is also required that the utility function be additively separable. Both assumptions allow us to write the consumer's problem as a two-moment decision model. Let ϵ be a random variable obtained from normalization of $c_2(\theta,s)$, that is, $\epsilon = \frac{(c_2-\mu_{c_2})}{\sigma_{c_2}}$ with cumulative distribution function $F(\epsilon)$ and support on $[a,b]$, where $\mu_{c_2} = \mu_{c_2}(\mu_\theta,s)$ and $\sigma_{c_2} = \sigma_{c_2}(\sigma_\theta,s)$ are the mean and standard deviation of $c_2(\theta,s)$ and the mean of the consumption in the first period is represented by $\mu_{c_1} = w - s$.

Thus, a risk-averse consumer has the following saving decision:

$$s^*(\mu_\theta,\sigma_\theta) = \underset{s}{argmax}\,\Phi(s) \tag{1a}$$

$$\Phi(s) = V(\mu_{c_1},0) + U(\mu_{c_2},\sigma_{c_2}) \tag{1b}$$

where

$$V(\mu_{c_1},0) \equiv v(w-s) \tag{2}$$

and

$$Eu(c_2) = \int_a^b u(\mu_{c_2} + \sigma_{c_2}\epsilon)dF(\epsilon) \equiv U(\mu_{c_2},\sigma_{c_2}) \tag{3}$$

Throughout the paper, we assume that the function U is strictly increasing in its first argument and is strictly decreasing and concave in the second: $U_\mu > 0 > U_\sigma$, $U_{\mu\mu} < 0$, $U_{\sigma\sigma} < 0$ and $U_{\mu\mu}U_{\sigma\sigma} - (U_{\mu\sigma})^2 > 0$ for all (μ_{c_2},σ_{c_2}). Furthermore, we assume that $V_{\mu\mu} < 0 < V_\mu$. The subscripts denote the partial derivatives, cross derivatives and second derivatives.

The positive marginal rate of substitution between μ and σ is defined by

$$R(\mu,\sigma) = -\frac{U_\sigma(\mu,\sigma)}{U_\mu(\mu,\sigma)} \tag{4}$$

This is a fundamental building block of this paper as, in the space of risk and return, it is common to interpret the marginal rate of substitution as a measure of risk aversion (Lajeri and Nielsen, 2000). The indifference curves in the (μ,σ)-approach are upward-sloping, and their slope is an indication of risk aversion. Thus, from hereon in, $R(\mu,\sigma)$ —as defined in (4)— will be referred to as the measure of risk aversion in the (μ,σ)-space.

[7] If consumers possess access to a perfect capital market, then consumption in the second period is linear in saving. However, consumers in developing countries cannot access a capital market and may only be able to invest in projects that offer decreasing returns to capital income, as is the case of on-farm investment. Then, consumption in the second period would be strictly concave (Gunning, 2010).

Given the assumptions of the model, the first- and second-order conditions of the maximization problem are as follows[8]:

$$\Phi_s = -V_\mu + U_\mu \frac{\partial \mu_{c_2}}{\partial s} + U_\sigma \frac{\partial \sigma_{c_2}}{\partial s} = 0 \tag{5}$$

$$\begin{aligned}\Phi_{ss} = V_{\mu\mu} + U_{\mu\mu}\left(\frac{\partial \mu_{c_2}}{\partial s}\right)^2 + 2U_{\mu\sigma}\frac{\partial \mu_{c_2}}{\partial s}\frac{\partial \sigma_{c_2}}{\partial s} \\ +U_{\sigma\sigma}\left(\frac{\partial \sigma_{c_2}}{\partial s}\right)^2 + U_\mu \frac{\partial^2 \mu_{c_2}}{\partial s^2} + U_\sigma \frac{\partial^2 \sigma_{c_2}}{\partial s^2} < 0\end{aligned} \tag{6}$$

Then, from the first-order condition, we have the following equation:

$$R(\mu_{c_2}, \sigma_{c_2}) = \frac{\partial \mu_{c_2}/\partial s - V_\mu/U_\mu}{\partial \sigma_{c_2}/\partial s} \tag{7}$$

This two-period consumption model is formulated so that all random consumption alternatives available to consumers are related to one another and to random variable θ only by location and scale parameters. So, no matter which utility function is specified in the expected utility space and no matter how θ is distributed,[9] the choice of the saving s under the expected utility framework can be represented as one which maximizes $\Phi(s)$, subject to the opportunity set in (μ, σ) space given by $\mu_{c_1} = w - s$, $\mu_{c_2} = \mu_{c_2}(\mu_\theta, s)$ and $\sigma_{c_2} = \sigma_{c_2}(\sigma_\theta, s)$.[10] Note that the optimal saving $s^*(\mu_\theta, \sigma_\theta)$ is a function of the μ_θ and σ_θ. We next begin to analyze the consequences of changes in μ_θ and σ_θ on savings.

4. Consequences of changes in μ_θ

Now we will study how a first-order increase in risk affects the optimal saving decision. One way to think of a first-order risk increase is to imagine a simple shift to the left of the probability density function of the distribution of results. This generates a new distribution that is first-order stochastically dominated by the original one. This second distribution has the same variance but a lower mean, so we can define a first-order risk increase as a reduction in the mean of distribution θ.

Intuitively, this comparative static exercise is very relevant for cases like the current worldwide pandemic. Since the coronavirus has unambiguously reduced the growth prospects of most nations, we can easily argue that the probability distribution of economic growth rates has moved dramatically to the left. We will analyze the changes in variance in the next section. For now, we will study how this change in first-order risk can be captured in the (μ, σ)-model in connection with the traditional expected utility of consumption paradigm. This idea is relevant because of its potential policy implications derived from government's attempts to help the economy to recover from the crisis generated by the pandemic.

The effect of changes in μ_θ on optimal saving is investigated by using the μ-related elasticity of risk aversion. The definition that we present below is based on Broll et al. (2006) and Eichner and Wagener (2014).

Definition 1. (Risk-aversion elasticity with respect to μ). Let $\rho_{\mu_{c_2}}$ denote the elasticity of risk aversion in relation to the mean of consumption in the second period. Then,

$$\rho_{\mu_{c_2}} = -\frac{\partial R}{\partial \mu_{c_2}}\frac{\mu_{c_2}}{R} \tag{8}$$

The term $\rho_{\mu_{c_2}}$ indicates the percentage change in risk aversion over the percentage change in the mean of consumption, holding the standard deviation of the consumption in the second period constant.[11]

Proposition 1. *Let $s^*(\mu_\theta, \sigma_\theta)$ be the optimal level of saving that maximizes $V(\mu_{c_1}, 0) + U(\mu_{c_2}, \sigma_{c_2})$. Then, s^* increases (decreases) when μ_θ decreases (increases) if and only if the following holds:*

$$\rho_{\mu_{c_2}} < (>) - \mu_{c_2}\left[\frac{\partial^2 \mu_{c_2}}{\partial s \partial \mu_\theta} + \frac{U_{\mu\sigma}}{U_\sigma}\frac{V_\mu}{U_\mu}\frac{\partial \mu_{c_2}}{\partial \mu_\theta}\right] / \frac{\partial \mu_{c_2}}{\partial s}\frac{\partial \mu_{c_2}}{\partial \mu_\theta}$$

where $\rho_{\mu_{c_2}}$ is the μ_{c_2}-related elasticity of risk aversion.[12]

Proof. Implicitly differentiating the first-order condition with respect to μ_θ yields

$$\frac{ds^*}{d\mu_\theta} = -\frac{\Phi_{s\mu_\theta}}{\Phi_{ss}} \tag{9}$$

Since the denominator corresponds to the second-order condition and is therefore negative, $\frac{ds^*}{d\mu_\theta}$ has the same sign as $\Phi_{s\mu_\theta}$. Differentiating Φ_s with respect to μ_θ yields

$$\Phi_{s\mu_\theta} = U_{\mu\mu}\frac{\partial \mu_{c_2}}{\partial s}\frac{\partial \mu_{c_2}}{\partial \mu_\theta} + U_\mu \frac{\partial^2 \mu_{c_2}}{\partial s \partial \mu_\theta} + U_{\mu\sigma}\frac{\partial \sigma_{c_2}}{\partial s}\frac{\partial \mu_{c_2}}{\partial \mu_\theta} \tag{10}$$

Since the relationship between consumption in the second period and the source of uncertainty is linear, a change in the mean of the shock (μ_θ) affects the mean of consumption in the second period (μ_{c_2}) but does not affect the standard deviation of consumption in the second period (σ_{c_2}). This permits us to connect equation (10) with the marginal rate of substitution defined in (4) and Definition 1. The change in μ_θ causes a disequilibrium, which is restored by modifying the optimal saving. Thus, this change in saving affects the mean and standard deviation of consumption in the second period, as well as the marginal rate of substitution $R(\mu, \sigma)$.

We now substitute $\frac{\partial \sigma_{c_2}}{\partial s}$ from (5) into (10) and rearrange the expression as follows:

$$\Phi_{s\mu_\theta} = \left[U_{\mu\mu} + \frac{U_{\mu\sigma}}{R}\right]\frac{\partial \mu_{c_2}}{\partial s}\frac{\partial \mu_{c_2}}{\partial \mu_\theta} + U_\mu \frac{\partial^2 \mu_{c_2}}{\partial s \partial \mu_\theta} + \frac{U_{\mu\sigma}V_\mu}{U_\sigma}\frac{\partial \mu_{c_2}}{\partial \mu_\theta} \tag{11}$$

To connect our results with risk-aversion elasticity, we differentiate $R(\mu_{c_2}, \sigma_{c_2})$ in relation to μ_{c_2}. At the optimum, the μ_{c_2}-related elasticity of risk aversion is given by

$$\rho_{\mu_{c_2}} \equiv -\frac{\partial R}{\partial \mu_{c_2}}\frac{\mu_{c_2}}{R} = \frac{\mu_{c_2}}{U_\mu}\left[U_{\mu\mu} + \frac{U_{\mu\sigma}}{R}\right] \tag{12}$$

Substituting (12) into (11) and reordering the expression yields

$$\Phi_{s\mu_\theta} = U_\mu\left[\frac{\rho_{\mu_{c_2}}}{\mu_{c_2}}\frac{\partial \mu_{c_2}}{\partial s}\frac{\partial \mu_{c_2}}{\partial \mu_\theta} + \frac{\partial^2 \mu_{c_2}}{\partial s \partial \mu_\theta} + \frac{U_{\mu\sigma}}{U_\sigma}\frac{V_\mu}{U_\mu}\frac{\partial \mu_{c_2}}{\partial \mu_\theta}\right] \tag{13}$$

Given that $U_\mu > 0$, $\Phi_{s\mu_\theta} < (>) 0$ whenever the expression in the square brackets is negative (positive). Therefore, $\frac{ds^*}{d\mu_\theta} < (>) 0$ if and only

[8] If the relationship between consumption and saving is linear, as we will see in the examples ahead, then $\frac{\partial^2 \mu_{c_2}}{\partial s^2} = \frac{\partial^2 \sigma_{c_2}}{\partial s^2} = 0$. Thus, the last two terms in condition (6) are equal to zero.

[9] Equation (3) defines the $U(\mu, \sigma)$ function associated with any utility function in expected utility space. Since no restriction has been placed on the form of utility function $u(\cdot)$ or cumulative distribution function $F(\cdot)$, substantial flexibility remains regarding the form that function $U(\mu, \sigma)$ can adopt. No quadratic utility or normally distributed random variables need to be assumed to reconcile the expected utility approach with the (μ, σ) approach.

[10] For example, let's assume that consumption in the second period is $c_2 = \theta y + rs$, where y is labor income, θ is a stochastic shock that affects labor income and r denotes the gross rate of interest. In this special case, the consumer maximizes $\Phi(s)$, subject to the opportunity set described by $\mu_{c_1} = w - s$, $\mu_{c_2} = \mu_\theta y + rs$ and $\sigma_{c_2} = \sigma_\theta y$.

[11] Broll et al. (2015) and Broll and Mukherjee (2017) define the mean-related elasticity of risk aversion as $\rho_\mu = \frac{\partial R}{\partial \mu}\frac{\mu}{R}$; this is without the minus sign. We use the minus sign in order to make our definition of elasticity intuitively comparable with the usual definition of elasticity from consumer theory.

[12] Note that $\rho_{\mu_\theta} = \rho_{\mu_{c_2}}\frac{\partial \mu_{c_2}}{\partial \mu_\theta}\frac{\mu_\theta}{\mu_{c_2}}$. Thus, the threshold for ρ_{μ_θ} is given by $-\mu_\theta\left[\frac{\partial^2 \mu_{c_2}}{\partial s \partial \mu_\theta} + \frac{U_{\mu\sigma}}{U_\sigma}\frac{V_\mu}{U_\mu}\frac{\partial \mu_{c_2}}{\partial \mu_\theta}\right] / \frac{\partial \mu_{c_2}}{\partial s}$.

if the following holds:

$$\rho_{\mu_{c_2}} < (>) - \mu_{c_2}[\frac{\partial^2 \mu_{c_2}}{\partial s \partial \mu_\theta} + \frac{U_{\mu\sigma}}{U_\sigma}\frac{V_\mu}{U_\mu}\frac{\partial \mu_{c_2}}{\partial \mu_\theta}] / \frac{\partial \mu_{c_2}}{\partial s}\frac{\partial \mu_{c_2}}{\partial \mu_\theta} \quad (14)$$

□

Note that the intertemporal nature of saving decisions means that the threshold at which the elasticity is compared is not unity or a constant. This point is exemplified in the next subsection, where we analyze the reduced form assumed by Proposition 1 for different types of risk.

4.1. Applications of changes in μ_θ and sources of risk

Gunning (2010) analyzed how four types of risk affected saving: labor income risk, wealth risk, asset risk and capital income risk. His work reasons that there may be a nonlinear relationship between saving and future consumption. This phenomenon occurs particularly in developing economies in which consumers must invest in projects with diminishing returns.[13]

Here, we analyze how the different types of risk defined in Gunning (2010) affect saving, in the context of first-order risk. In the (μ,σ)-space, this means a shift in the mean of the shock (μ_θ) that affects consumption in the second period.

Gunning establishes the relationship between consumption in the second period and labor income risk through the following equation:

$$c_2(\theta, s) = \theta y + (1-\delta)s + h(s) \quad (15)$$

where y is the expected labor income, $(1-\delta)s$ is the expected value of physical assets, $h(s)$ is the expected value of capital income and δ is the depreciation rate ($0 < \delta < 1$). The function $h(s)$ is increasing and concave, with $h(0) = 0$. In this case, the mean of c_2 is $\mu_{c_2} = \mu_\theta y + (1-\delta)s + h(s)$. Thus, $\frac{\partial \mu_{c_2}}{\partial s} = (1-\delta) + h'(s)$, $\frac{\partial \mu_{c_2}}{\partial \mu_\theta} = y$, and $\frac{\partial^2 \mu_{c_2}}{\partial s \partial \mu_\theta} = 0$. Applying Proposition 1, we have that $\frac{ds^*}{d\mu_\theta} < (>)0$ if and only if $\rho_{\mu_{c_2}} < (>) - \frac{\mu_{c_2}}{(1-\delta)+h'(s)}\frac{U_{\mu\sigma}}{U_\sigma}\frac{V_\mu}{U_\mu}$.

For the case of wealth risk, Gunning establishes the following relationship:

$$c_2(\theta, s) = \theta(y + (1-\delta)s + h(s)) \quad (16)$$

In this case, the mean is $\mu_{c_2} = \mu_\theta(y + (1-\delta)s + h(s))$. Following the same procedure as above, applying Proposition 1, we have that $\frac{ds^*}{d\mu_\theta} < (>)0$ if and only if

$$\rho_{\mu_{c_2}} < (>) - [1 + \frac{\mu_{c_2}}{\mu_\theta((1-\delta)+h'(s))}\frac{U_{\mu\sigma}}{U_\sigma}\frac{V_\mu}{U_\mu}]$$

If the shock that affects consumption is to assets, then consumption in the second period is

$$c_2(\theta, s) = y + \theta(1-\delta)s + h(s) \quad (17)$$

The mean consumption for this type of risk is $\mu_{c_2} = y + \mu_\theta(1-\delta)s + h(s)$. According to Proposition 1, the threshold for an increase in the mean is determined by the relationship $\frac{ds^*}{d\mu_\theta} < (>)0$ if and only if $\rho_{\mu_{c_2}} < (>) - \frac{\mu_{c_2}}{\mu_{c_2}-b}[1 + \frac{U_{\mu\sigma}}{U_\sigma}\frac{V_\mu}{U_\mu}s]$, where $b = y + h(s) - h'(s)s$.

Finally, capital income risk is represented by the following equation:

$$c_2(\theta, s) = y + (1-\delta)s + \theta h(s) \quad (18)$$

where the mean is $\mu_{c_2} = y + (1-\delta)s + \mu_\theta h(s)$. For this type of risk, saving increases (decreases) if and only if $\rho_{\mu_{c_2}} < (>) - \frac{\mu_{c_2}}{\mu_{c_2}-b}[1 + \frac{U_{\mu\sigma}}{U_\sigma}\frac{V_\mu}{U_\mu}\frac{h(s)}{h'(s)}]$, where $b = y + (1-\delta)[s - \frac{h(s)}{h'(s)}]$. We note that for $\delta = 1$ and $h(s) = rs$, consumption in the second period is $c_2 = y + \theta rs$, the mean is $\mu_{c_2} = y + \mu_\theta rs$ and $b = y$, which was the analysis made by Eeckhoudt and Schlesinger (2008) for interest rate risk. Here, the threshold is determined by the relationship $\rho_{\mu_{c_2}} < (>) - \frac{\mu_{c_2}}{\mu_{c_2}-y}[1 + \frac{U_{\mu\sigma}}{U_\sigma}\frac{V_\mu}{U_\mu}s]$.

As we have seen in this subsection, because of intertemporal structural preferences, the uncertainty adds complexity to the consumer's problem. The second element in the numerator of the threshold in condition (14), i.e., $\frac{U_{\mu\sigma}}{U_\sigma}\frac{V_\mu}{U_\mu}\frac{\partial \mu_{c_2}}{\partial \mu_\theta}$, is a function that depends on saving. Thus, such an element makes the threshold not equal to unity or a constant.

In the next subsection, we analyze the consumer's problem in the expected utility space under first-order risk. We derive two equivalent general indices of risk aversion and apply them to the different types of risk described above. The first index is derived using the method developed by Sinn (1983) and Meyer (1987), while the second index is derived following Diamond and Stiglitz (1974). We show the equivalence between the indices and the μ-related risk aversion elasticity.

4.2. Comparison with the expected utility framework

It is common knowledge that the (μ,σ)-framework is equivalent to the expected utility approach if all attainable lotteries belong to the same linear family. This holds in the consumer's problem, as the relationship between saving s and shock θ that affects income in the second period is linear and the utility function is additively separable.

To connect our results in the (μ,σ)-space with the expected utility approach, following Sinn (1983) and Meyer (1987), we establish the equivalences between $u(\cdot)$ and $U(\cdot,\cdot)$ as follows:

$$U_\mu(\mu_{c_2}, \sigma_{c_2}) = \int_a^b u'(\mu_{c_2} + \sigma_{c_2}\epsilon)dF(\epsilon) > 0 \quad (19a)$$

$$U_{\mu\mu}(\mu_{c_2}, \sigma_{c_2}) = \int_a^b u''(\mu_{c_2} + \sigma_{c_2}\epsilon)dF(\epsilon) < 0 \quad (19b)$$

$$U_{\mu\sigma}(\mu_{c_2}, \sigma_{c_2}) = \int_a^b u''(\mu_{c_2} + \sigma_{c_2}\epsilon)\epsilon dF(\epsilon) > 0 \quad (19c)$$

Proposition 2. *Let $s^*(\mu_\theta, \sigma_\theta)$ be the optimal level of saving that maximizes $V(\mu_{c_1}, 0) + U(\mu_{c_2}, \sigma_{c_2})$. Then, s^* increases (decreases) when μ_θ decreases if and only if the following holds:*

$A(c_2) > (<) \frac{\partial^2 \mu_{c_2}}{\partial s \partial \mu_\theta} / \frac{\partial c_2}{\partial s}\frac{\partial \mu_{c_2}}{\partial \mu_\theta}$, where $A(c_2) = -u''(c_2)/u'(c_2)$ is the absolute aversion coefficient.

Proof. Substituting (19a), (19b) and (19c) into (10) yields

$$\Phi_{s\mu_\theta} = \int_a^b [u''(c_2)\frac{\partial c_2}{\partial s}\frac{\partial \mu_{c_2}}{\partial \mu_\theta} + u'(c_2)\frac{\partial^2 \mu_{c_2}}{\partial s \partial \mu_\theta}]dF(\epsilon) \quad (20)$$

Then, condition (20) is negative (positive) whenever the following holds:

$$\Phi_{s\mu_\theta} = u''(c_2)\frac{\partial c_2}{\partial s}\frac{\partial \mu_{c_2}}{\partial \mu_\theta} + u'(c_2)\frac{\partial^2 \mu_{c_2}}{\partial s \partial \mu_\theta} < (>)0 \quad (21)$$

By rearranging the above expression, we obtain Proposition 2.

□

Let $F(\theta, \tau)$ be the cumulative distribution function of θ, defined over the support within the closed interval $[a, b]$, where τ is a parameter whose shift represents an increase in risk, as defined by Diamond and Stiglitz (1974). Consider two distributions $F(\theta, \tau_1)$ and $F(\theta, \tau_2)$. If $F(\theta, \tau_2) - F(\theta, \tau_1) \geq 0 \forall \theta \in [a, b]$ with strong inequality for any θ, then $F(\theta, \tau_1)$ dominates $F(\theta, \tau_2)$ via first-order stochastic dominance, which is clearly the same as $F(\theta, \tau_2)$ is a first-order risk increase of $F(\theta, \tau_1)$. When τ is a continuous variable, the definition of first-order increase in risk is $F_\tau \geq 0$. We use this last definition to connect the relationship

[13] Gunning's formulation is based on a CRRA (constant relative risk aversion) utility function and a mean-preserving increase in risk.

between saving and a shift in τ. From this relationship, we derive a general risk aversion index equivalent to that obtained in Proposition 2.

In the expected utility space, a risk-averse consumer's saving decision is defined by

$$s^*(\tau) = \underset{s}{argmax}\,\Psi(s) \tag{22a}$$

$$\Psi(s) = v(c_1) + Eu(c_2) = v(c_1) + \int_a^b u(c_2(\theta, s))dF(\theta, \tau) \tag{22b}$$

The first-and second-order conditions are[14]

$$\Psi_s = -v'(w - s^*) + \int_a^b u'(c_2)\frac{\partial c_2}{\partial s}dF(\theta, \tau) = 0 \tag{23}$$

$$\Psi_{ss} = v''(w - s^*) + \int_a^b [u''(c_2)(\frac{\partial c_2}{\partial s})^2 + u'(c_2)\frac{\partial^2 c_2}{\partial s^2}]dF(\theta, \tau) < 0 \tag{24}$$

Proposition 3. *Let $s^*(\tau)$ be the optimal level of saving that maximizes $v(c_1) + \int_a^b u(c_2)\,dF(\theta, \tau)$. If a shift in τ represents a first-order risk increase, then s^* increases (decreases) if and only if the following holds:*

$A(c_2) > (<) \frac{\partial^2 c_2}{\partial s \partial \theta} / \frac{\partial c_2}{\partial s}\frac{\partial c_2}{\partial \theta}$, where $A(c_2) = -u''(c_2)/u'(c_2)$ is the absolute aversion coefficient.

Proof. Implicitly differentiating the first-order condition with respect to τ yields

$$\frac{ds^*}{d\tau} = -\frac{\Psi_{s\tau}}{\Psi_{ss}} = -\frac{\int_a^b u'(c_2)\frac{\partial c_2}{\partial s}dF_\tau(\theta, \tau)}{\Psi_{ss}} \tag{25}$$

Since the denominator corresponds to the second-order condition and is therefore negative, $\frac{ds^*}{d\tau}$ has the same sign as the numerator. Applying integration by parts once, given that $F_\tau(a, \tau) = F_\tau(b, \tau) = 0$, we obtain

$$\Psi_{s\tau} = \int_a^b u'(c_2)\frac{\partial c_2}{\partial s}dF_\tau(\theta, \tau) = -\int_a^b [u''(c_2)\frac{\partial c_2}{\partial s}\frac{\partial c_2}{\partial \theta} + u'(c_2)\frac{\partial^2 c_2}{\partial s \partial \theta}]F_\tau d\theta \tag{26}$$

Given that our relevant comparative static is a first-order risk increase, then $F_\tau \geq 0$ and we can directly show that $\frac{ds^*}{d\tau} > (<)0$ if and only if the following holds:

$$u''(c_2)\frac{\partial c_2}{\partial s}\frac{\partial c_2}{\partial \theta} + u'(c_2)\frac{\partial^2 c_2}{\partial s \partial \theta} < (>)0 \tag{27}$$

which implies

$$A(c_2) > (<)\frac{\partial^2 c_2}{\partial s \partial \theta} / \frac{\partial c_2}{\partial s}\frac{\partial c_2}{\partial \theta} \tag{28}$$

□

The above analysis allows us to conclude that $\frac{ds^*}{d\mu_\theta} < (>)0$ is equivalent to $\frac{ds^*}{d\tau} > (<)0$, under the assumptions established in this paper, i.e., linear risks and intertemporally separable preferences. For example, we consider a labor income risk. In the (μ, σ)-space, equation (10) is reduced to $\Phi_{s\mu_\theta} = U_{\mu\mu}y((1 - \delta) + h'(s))$. Thus, $\frac{ds^*}{d\mu_\theta} < 0$ if and only if $U_{\mu\mu} < 0$, which from (21) is equivalent to $u'' < 0$. In other words, a consumer increases saving when the mean of the shock that affects income in the second period is decreased, if and only if the consumer is risk averse.

For wealth risk, the index given in (28) is transformed to $c_2 A(c_2) > (<)1$, while for asset risk and capital income risk the general index is reduced to $(c_2 - b)A(c_2) > (<)1$, where b is described above and $(c_2 - b)A(c_2)$ is a special case of the index of partial relative risk aversion introduced by Menezes and Hanson (1970) and Zeckhauser and Keeler (1970). In the case of interest rate risk, the threshold is determined by $(c_2 - y)A(c_2) > (<)1$.[15]

Thus, in the context of first-order risk, Table 1 summarizes the threshold for the five types of risk.

5. Consequences of changes in σ_θ

We now analyze the relationship between saving and a change in the standard deviation of θ. The relationship is investigated by using the σ-related elasticity of risk aversion. The definition that we present below is based on Battermann et al. (2002), Eichner and Wagener (2004a, 2004b) and Broll et al. (2006).

Definition 2. (Risk aversion elasticity with respect to σ)**.** Let $\rho_{\sigma_{c_2}}$ denote the elasticity of risk aversion with respect to the standard deviation of consumption in the second period. Then,

$$\rho_{\sigma_{c_2}} = -\frac{\partial R}{\partial \sigma_{c_2}}\frac{\sigma_{c_2}}{R} \tag{29}$$

$\rho_{\sigma_{c_2}}$ indicates the percentage change in risk aversion over the percentage change in the standard deviation of consumption, holding the mean of the consumption in the second period constant.[16]

Proposition 4. *Let $s^*(\mu_\theta, \sigma_\theta)$ be the optimal level of saving that maximizes $V(\mu_{c_1}, 0) + U(\mu_{c_2}, \sigma_{c_2})$. Then, s^* increases (decreases) when σ_θ increases if and only if the following holds:*

$$\rho_{\sigma_{c_2}} > (<)\sigma_{c_2}[\frac{\partial^2 \sigma_{c_2}}{\partial \sigma_\theta \partial s} + \frac{U_{\mu\sigma}}{U_\sigma}\frac{V_\mu}{U_\mu}\frac{\partial \sigma_{c_2}}{\partial \sigma_\theta}] / \frac{\partial \sigma_{c_2}}{\partial s}\frac{\partial \sigma_{c_2}}{\partial \sigma_\theta}$$

where $\rho_{\sigma_{c_2}}$ is the elasticity of risk aversion with respect to σ_{c_2}.[17]

Proof. Implicitly differentiating the first-order condition (5) with respect to σ_θ, we obtain

$$\frac{ds^*}{d\sigma_\theta} = -\frac{\Phi_{s\sigma_\theta}}{\Phi_{ss}} \tag{30}$$

Since the denominator is the second order condition and therefore negative, $\frac{ds^*}{d\sigma_\theta}$ has the same sign as the numerator. Differentiating Φ_s with respect to σ_θ yields

$$\Phi_{s\sigma_\theta} = U_{\mu\sigma}\frac{\partial \mu_{c_2}}{\partial s}\frac{\partial \sigma_{c_2}}{\partial \sigma_\theta} + U_{\sigma\sigma}\frac{\partial \sigma_{c_2}}{\partial s}\frac{\partial \sigma_{c_2}}{\partial \sigma_\theta} + U_\sigma\frac{\partial^2 \sigma_{c_2}}{\partial \sigma_\theta \partial s} \tag{31}$$

As the source of uncertainty is linear, a change in the standard deviation of the shock (σ_θ) affects the standard deviation of the consumption in the second period (σ_{c_2}) but does not affect its expected value μ_{c_2}. This allows us to connect equation (31) with the marginal rate of substitution $R(\mu, \sigma)$ and Definition 2. Then, the shift in σ_θ modifies saving and with it the mean and standard deviation of consumption in the second period, as well as the marginal rate of substitution between μ and σ.

[14] If the relationship between consumption and saving is linear, then $\frac{\partial^2 c_2}{\partial s^2} = 0$. Thus, the second term within the square parenthesis in condition (24) is equal to zero.

[15] Eeckhoudt and Schlesinger (2008) consider the special case when $y = 0$. Thus, an increase in interest rate risk generates an increase in saving if the index of relative risk aversion is greater than one. Additionally, Fishburn and Porter (1976) showed that an increase in the interest rate risk in a portfolio problem increases the saving if the partial relative risk aversion index is greater than one, which is consistent with our results.

[16] Broll et al. (2015) and Broll and Mukherjee (2017) define the standard deviation-related elasticity of risk aversion as $\rho_\sigma = \frac{\partial R}{\partial \sigma}\frac{\sigma}{R}$; this is without the minus sign. We use the minus sign in order to make our definition of elasticity intuitively comparable with the usual definition of elasticity from consumer theory.

[17] Note that $\rho_{\sigma_\theta} = \rho_{\sigma_{c_2}}\frac{\partial \sigma_{c_2}}{\partial \sigma_\theta}\frac{\sigma_\theta}{\sigma_{c_2}}$. Thus, it can be directly shown that the threshold for ρ_{σ_θ} is given by $\sigma_\theta[\frac{\partial^2 \sigma_{c_2}}{\partial \sigma_\theta \partial s} + \frac{U_{\mu\sigma}}{U_\sigma}\frac{V_\mu}{U_\mu}\frac{\partial \sigma_{c_2}}{\partial \sigma_\theta}] / \frac{\partial \sigma_{c_2}}{\partial s}$.

Table 1
First-order risk: μ_θ versus τ.

Type of risk	$\frac{ds^*}{d\mu_\theta}$	$\frac{ds^*}{d\tau}$	General risk aversion index	Elasticity of risk aversion
Labor income risk	$-(+)$	$+(-)$	$u''(c_2) < (>)0$	$\rho_{\mu_{c_2}} < (>) - \frac{\mu_{c_2}}{(1-\delta)+h'(s)} \frac{U_{\mu\sigma}}{U_\sigma} \frac{V_\mu}{U_\mu}$
Wealth risk	$-(+)$	$+(-)$	$c_2 A(c_2) > (<)1$	$\rho_{\mu_{c_2}} < (>) - [1 + \frac{\mu_{c_2}}{\mu_\theta((1-\delta)+h'(s))} \frac{U_{\mu\sigma}}{U_\sigma} \frac{V_\mu}{U_\mu}]$
Asset risk	$-(+)$	$+(-)$	$(c_2 - b)A(c_2) > (<)1$	$\rho_{\mu_{c_2}} < (>) - \frac{\mu_{c_2}}{\mu_{c_2}-b}[1 + \frac{U_{\mu\sigma}}{U_\sigma} \frac{V_\mu}{U_\mu} s]$
Capital income risk	$-(+)$	$+(-)$	$(c_2 - b)A(c_2) > (<)1$	$\rho_{\mu_{c_2}} < (>) - \frac{\mu_{c_2}}{\mu_{c_2}-b}[1 + \frac{U_{\mu\sigma}}{U_\sigma} \frac{V_\mu}{U_\mu} \frac{h(s)}{h'(s)}]$
Interest rate risk	$-(+)$	$+(-)$	$(c_2 - y)A(c_2) > (<)1$	$\rho_{\mu_{c_2}} < (>) - \frac{\mu_{c_2}}{\mu_{c_2}-y}[1 + \frac{U_{\mu\sigma}}{U_\sigma} \frac{V_\mu}{U_\mu} s]$

We now substitute $\frac{\partial \mu_{c_2}}{\partial s}$ from (5) into (31) and rearrange the expression as follows:

$$\Phi_{s\sigma_\theta} = \{U_{\mu\sigma}\frac{V_\mu}{U_\mu} + [U_{\mu\sigma}R + U_{\sigma\sigma}]\frac{\partial \sigma_{c_2}}{\partial s}\}\frac{\partial \sigma_{c_2}}{\partial \sigma_\theta} + U_\sigma \frac{\partial^2 \sigma_{c_2}}{\partial \sigma_\theta \partial s} \tag{32}$$

To connect our results to the elasticity of risk aversion, we differentiate $R(\mu_{c_2}, \sigma_{c_2})$ with respect to σ_{c_2}. At the optimum, the σ_{c_2}-related elasticity of risk aversion is given by

$$\rho_{\sigma_{c_2}} \equiv -\frac{\partial R}{\partial \sigma_{c_2}}\frac{\sigma_{c_2}}{R} = -\frac{\sigma_{c_2}}{U_\sigma}[U_{\sigma\sigma} + RU_{\mu\sigma}] \tag{33}$$

Substituting (33) into (32) and reordering the expression yields

$$\Phi_{s\sigma_\theta} = U_\sigma[\frac{U_{\mu\sigma}}{U_\sigma}\frac{V_\mu}{U_\mu}\frac{\partial \sigma_{c_2}}{\partial \sigma_\theta} - \frac{\rho_{\sigma_{c_2}}}{\sigma_{c_2}}\frac{\partial \sigma_{c_2}}{\partial s}\frac{\partial \sigma_{c_2}}{\partial \sigma_\theta} + \frac{\partial^2 \sigma_{c_2}}{\partial \sigma_\theta \partial s}] \tag{34}$$

Given that $U_\sigma < 0$, $\Phi_{s\sigma_\theta} > (<)0$ whenever the expression in the square brackets is negative (positive). Therefore, $\frac{ds^*}{d\sigma_\theta} > (<)0$ if and only if the following holds:

$$\rho_{\sigma_{c_2}} > (<)\sigma_{c_2}[\frac{\partial^2 \sigma_{c_2}}{\partial \sigma_\theta \partial s} + \frac{U_{\mu\sigma}}{U_\sigma}\frac{V_\mu}{U_\mu}\frac{\partial \sigma_{c_2}}{\partial \sigma_\theta}]/\frac{\partial \sigma_{c_2}}{\partial s}\frac{\partial \sigma_{c_2}}{\partial \sigma_\theta} \tag{35}$$

□

In the next subsection we describe the form assumed by Proposition 4 for the different sources of risk.

5.1. Applications of changes in σ_θ and sources of risk

Here, we study the effects of the different sources of risk defined in Gunning (2010) on saving. In the (μ, σ)-space, a second-order risk increase translates into an increase in the standard deviation of the shock that affects consumption in the second period.

For the case of labor income risk given in equation (15), the standard deviation is $\sigma_{c_2} = \sigma_\theta y$. Thus, $\frac{\partial \sigma_{c_2}}{\partial \sigma_\theta} = y$, and $\frac{\partial \sigma_{c_2}}{\partial s} = \frac{\partial^2 \sigma_{c_2}}{\partial \sigma_\theta \partial s} = 0$. Note that, for this type of risk, there is no elasticity of risk aversion. Let us remember that $\rho_{\sigma_{c_2}}$ measures the percentage change in the marginal rate of substitution over the percentage change in the standard deviation of c_2. From the first-order condition, we know that $R(\mu_{c_2}, \sigma_{c_2}) = (\frac{\partial \mu_{c_2}}{\partial s} - \frac{V_\mu}{U_\mu})/\frac{\partial \sigma_{c_2}}{\partial s}$. Given that $\frac{\partial \sigma_{c_2}}{\partial s} = 0$, there is no marginal rate of substitution. Therefore, there is no elasticity because the marginal rate of substitution is infinite.

When the shock is to wealth, as described in equation (16), the standard deviation of consumption in the second period is $\sigma_{c_2} = \sigma_\theta(y + (1-\delta)s + h(s))$. Applying Proposition 4, we have that $\frac{ds^*}{d\sigma_\theta} > (<)0$ if and only if $\rho_{\sigma_{c_2}} > (<)[1 + \frac{\sigma_{c_2}}{\sigma_\theta((1-\delta)+h'(s))}\frac{U_{\mu\sigma}}{U_\sigma}\frac{V_\mu}{U_\mu}]$.

As for asset risk and capital income risk, the standard deviations of consumption are $\sigma_{c_2} = \sigma_\theta(1-\delta)s$ and $\sigma_{c_2} = \sigma_\theta h(s)$, respectively. Applying Proposition 4 for asset risk, saving increases (decreases) if and only if $\rho_{\sigma_{c_2}} > (<)[1 + \frac{U_{\mu\sigma}}{U_\sigma}\frac{V_\mu}{U_\mu}s]$. On the other hand, for capital income risk, saving increases (decreases) if and only if $\rho_{\sigma_{c_2}} > (<)[1 + \frac{U_{\mu\sigma}}{U_\sigma}\frac{V_\mu}{U_\mu}\frac{h(s)}{h'(s)}]$. For the special case of interest rate risk, saving increases (decreases) if and only if $\rho_{\sigma_{c_2}} > (<)[1 + \frac{U_{\mu\sigma}}{U_\sigma}\frac{V_\mu}{U_\mu}s]$.

Similar to the analysis in subsection 4.2, we now analyze the consumer's problem in the expected utility space under second-order risk. First, we derive a general prudence index using the method developed by Sinn (1983) and Meyer (1987). Then, we derive an equivalent index using the method developed by Diamond and Stiglitz (1974). We apply both indices to the different types of risk described above and show the equivalence with the thresholds determined for the σ-related elasticity of risk aversion.

5.2. Comparison with the expected utility framework

In the context of precautionary saving and in line with Sinn (1983) and Meyer (1987), we establish the relationship between $u(\cdot)$ and $U(\cdot,\cdot)$ through two more properties in addition to those already proposed in subsection 4.2:

$$U_\sigma(\mu_{c_2}, \sigma_{c_2}) = \int_a^b u'(\mu_{c_2} + \sigma_{c_2}\epsilon)\epsilon dF(\epsilon) < 0 \tag{36a}$$

$$U_{\sigma\sigma}(\mu_{c_2}, \sigma_{c_2}) = \int_a^b u''(\mu_{c_2} + \sigma_{c_2}\epsilon)\epsilon^2 dF(\epsilon) < 0 \tag{36b}$$

Proposition 5. *Let $s^*(\mu_\theta, \sigma_\theta)$ be the optimal level of saving that maximizes $V(\mu_{c_1}, 0) + U(\mu_{c_2}, \sigma_{c_2})$. Then, s^* increases (decreases) when σ_θ increases if and only if the following holds:*

$$P(c_2) > (<)(\frac{\partial \sigma_{c_2}}{\partial s}\frac{\partial \sigma_{c_2}}{\partial \sigma_\theta} + \sigma_{c_2}\frac{\partial^2 \sigma_{c_2}}{\partial \sigma_\theta \partial s})/(\sigma_{c_2}\frac{\partial c_2}{\partial s}\frac{\partial \sigma_{c_2}}{\partial \sigma_\theta})$$

where $P(c_2) = -u'''(c_2)/u''(c_2)$ is the absolute prudence coefficient.

Proof. Substituting (36a), (36b) and (19c) into (31) yields

$$\Phi_{s\sigma_\theta} = \int_a^b [u''(c_2)\frac{\partial c_2}{\partial s}\frac{\partial \sigma_{c_2}}{\partial \sigma_\theta} + u'(c_2)\frac{\partial^2 \sigma_{c_2}}{\partial \sigma_\theta \partial s}]\epsilon dF(\epsilon) \tag{37}$$

Applying integration by parts, we obtain

$$\Phi_{s\sigma_\theta} = -\int_a^b (\int_a^\epsilon z dF(z)[u'''(c_2)\frac{\partial c_2}{\partial s}\frac{\partial \sigma_{c_2}}{\partial \sigma_\theta}\sigma_{c_2} + u''(c_2)(\frac{\partial \sigma_{c_2}}{\partial s}\frac{\partial \sigma_{c_2}}{\partial \sigma_\theta} + \frac{\partial^2 \sigma_{c_2}}{\partial \sigma_\theta \partial s}\sigma_{c_2})])d\epsilon \tag{38}$$

Since $E(\epsilon) = 0$, condition (38) is positive (negative) whenever the following holds:

$$u'''(c_2)\frac{\partial c_2}{\partial s}\frac{\partial \sigma_{c_2}}{\partial \sigma_\theta}\sigma_{c_2} + u''(c_2)(\frac{\partial \sigma_{c_2}}{\partial s}\frac{\partial \sigma_{c_2}}{\partial \sigma_\theta} + \frac{\partial^2 \sigma_{c_2}}{\partial \sigma_\theta \partial s}\sigma_{c_2}) > (<)0 \tag{39}$$

Rearranging the above expression, we have that $ds^*/d\sigma_\theta > (<)0$ if and only if the following holds:

$$P(c_2) > (<)(\frac{\partial \sigma_{c_2}}{\partial s}\frac{\partial \sigma_{c_2}}{\partial \sigma_\theta} + \sigma_{c_2}\frac{\partial^2 \sigma_{c_2}}{\partial \sigma_\theta \partial s})/(\sigma_{c_2}\frac{\partial c_2}{\partial s}\frac{\partial \sigma_{c_2}}{\partial \sigma_\theta}) \tag{40}$$

□

Next, we derive the general prudence index equivalent to that obtained in Proposition 5 using the concept of a mean-preserving spread.

Diamond and Stiglitz (1974) state that an increase in τ represents a mean-preserving increase in risk if the following two hold:

$$\int_a^b F_\tau(\theta,\tau)d\theta = 0 \tag{41}$$

and

$$T(\theta,\tau) = \int_a^\theta F_\tau(z,\tau)dz \geq 0 \quad \text{for all} \quad a \leq \theta \leq b \tag{42}$$

The discrete version of such a definition is as follows. Consider two distributions $F(\theta,\tau_1)$ and $F(\theta,\tau_2)$; then, conditions (41) and (42) can be expressed as

$$\int_a^b [F(\theta,\tau_2) - F(\theta,\tau_1)]d\theta = 0 \tag{43}$$

and

$$\int_a^\theta [F(z,\tau_2) - F(z,\tau_1)]dz \geq 0 \quad \text{for all} \quad a \leq \theta \leq b \tag{44}$$

The first condition means that the two distributions have the same mean, while the second is the so-called single-crossing property of the mean-preserving spread (Rothschild and Stiglitz, 1970).

Proposition 6. *Let $s^*(\tau)$ be the optimal level of saving that maximizes $v(c_1) + \int_a^b u(c_2)dF(\theta,\tau)$. If a shift in τ represents a mean-preserving spread, then s^* increases (decreases) if and only if the following holds:*

$P(c_2) > (<)2\frac{\partial^2 c_2}{\partial s\partial\theta}/\frac{\partial c_2}{\partial s}\frac{\partial c_2}{\partial\theta}$, where $P(c_2) = -u'''(c_2)/u''(c_2)$ is the absolute prudence coefficient.

Proof. From (25), we know that $\frac{ds^*}{d\tau}$ has the same sign as the numerator, which is represented by $\Psi_{s\tau}$. Applying integration by parts twice, and given that $F_\tau(a,\tau) = F_\tau(b,\tau) = T(a,\tau) = T(b,\tau) = 0$, we obtain the following result:

$$\begin{aligned}\Psi_{s\tau} &= \int_a^b u'(c_2)\frac{\partial c_2}{\partial s}dF_\tau(\theta,\tau) \\ &= \int_a^b [u'''(c_2)\frac{\partial c_2}{\partial s}\frac{\partial c_2}{\partial\theta} + 2u''(c_2)\frac{\partial^2 c_2}{\partial s\partial\theta}]\frac{\partial c_2}{\partial\theta}T(\theta,\tau)d\theta\end{aligned} \tag{45}$$

Given that $T(\theta,\tau) \geq 0$ and $\frac{\partial c_2}{\partial\theta} > 0$, the optimal saving s^* increases (decreases) if and only if the following holds:

$$u'''(c_2)\frac{\partial c_2}{\partial s}\frac{\partial c_2}{\partial\theta} + 2u''(c_2)\frac{\partial^2 c_2}{\partial s\partial\theta} > (<)0 \tag{46}$$

which implies

$$P(c_2) > (<)2\frac{\partial^2 c_2}{\partial s\partial\theta}/\frac{\partial c_2}{\partial s}\frac{\partial c_2}{\partial\theta} \tag{47}$$

□

This index is precisely that derived by Vergara (2017),[18] which is equivalent to that obtained in Proposition 5. Note that, for the case of labor income risk, condition (47) is reduced to $u''' > (<)0$. Therefore, prudence $u'''(>0)$ will guarantee that the precautionary effect is positive following an increase in labor income risk. In this case, a consumer increases saving when faced with riskier future labor income (in terms of the mean-preserving spread) if and only if u' is convex. This result is achieved by applying Proposition 5. We know from section 4.1 that, for labor income risk, $\frac{\partial\sigma_{c_2}}{\partial s} = \frac{\partial^2\sigma_{c_2}}{\partial\sigma_\theta\partial s} = 0$. Thus, the general prudence index given in (40) is reduced to $u''' > (<)0$. Note also that equation (31) is reduced to $\Phi_{s\sigma_\theta} = U_{\mu\sigma}y((1-\delta)+h'(s))$. Then, $\frac{ds^*}{d\sigma_\theta} > 0$ if and only if $U_{\mu\sigma} > 0$, which is equivalent to $u''' > 0$ (Meyer, 1987; Wagener, 2002). As we see, the threshold for saving with an increase in labor income risk is determined by $u''' > 0$, which has its analogy in the (μ,σ)-space of $U_{\mu\sigma} > 0$. However, the threshold cannot be obtained through the standard deviation-related elasticity of risk aversion. Thus, the elasticity of risk aversion does not connect with the concept of prudence defined by Kimball (1990).

When wealth risk, asset risk, capital income risk or interest rate risk conform sources of uncertainty, prudence can no longer guarantee the threshold that determines the direction of the precautionary effect. For the special case of wealth risk, both indices are transformed to $c_2P(c_2) > (<)2$; i.e., a relative prudence greater than 2 guarantees that precautionary saving occurs. For asset risk and capital income risk, the general indices are reduced to $(c_2 - b)P(c_2) > (<)2$. Thus, in these cases, saving increases if and only if the measure of partial prudence exceeds 2, where b is described above and $(c_2 - b)P(c_2)$ is a special case of the index of partial relative prudence introduced by Choi et al. (2001). In the special case of interest rate risk, the threshold is $(c_2 - y)P(c_2) > (<)2$.[19]

The analysis above allows us to conclude that $\frac{ds^*}{d\sigma_\theta} > (<)0$ is equivalent to $\frac{ds^*}{d\tau} > (<)0$ when a shift in τ represents a mean-preserving increase in risk. In the context of second-order risk, Table 2 summarizes the thresholds for the five types of risk.

6. Connecting our results with other applications in economics

In this section, we relate the results of our paper with some economics decision problems analyzed under the (μ,σ)-approach. In the context of first-order risk, we show that, in all cases, the threshold is determined by the following relationship: the control variable increases (decreases) when the mean of the shock decreases if and only if $\rho_{\mu_i} < (>) -\frac{\mu_i}{\mu_i - b}$, which is equivalent to $(i - b)A(i) > (<)1$, where $A(i) = -u''(i)/u'(i)$ is the absolute risk aversion coefficient. Under second-order risk, we show that, in all cases, the threshold is determined by the following relationship: the control variable increases (decreases) when the standard deviation of the shock increases if and only if $\rho_{\sigma_i} > (<)1$, which is equivalent to $(i - b)P(i) > (<)2$, where $P(i) = -u'''(i)/u''(i)$ is the absolute prudence coefficient. Thus, none of the cases show the presence of the term that accounts for the intertemporal nature of preferences in our model and that is part of the threshold in the elasticity of risk aversion.

Broll et al. (2006) analyzed the relationship between the elasticity of risk aversion and international trade. In this paper, $\pi = epq - C(q)$ denotes an exporting firm's risky profit, considering a random foreign exchange rate e, and where p is the commodity price expressed in a foreign currency and q is the quantity of a final good produced by the firm. In this case, the mean (for a given q) is $\mu_\pi = \mu_e pq - C(q)$, while the standard deviation is $\sigma_\pi = \sigma_e pq$. Applying Proposition 1, we obtain that $\frac{dq^*}{d\mu_e} < (>)0$ if and only if $\rho_{\mu_\pi} < (>) -\frac{\mu_\pi}{\mu_\pi - b} = -\frac{1}{R\sigma_\pi/\mu_\pi}$, where $R = -\frac{U_\sigma}{U_\mu}$ and $b = q[C'(q) - \frac{C(q)}{q}]$. This result is established in Proposition 2 of Broll et al. (2006). Using Proposition 2 of our paper, such a result is equivalent to $(\pi - b)A(\pi) > (<)1$. In the context of second-order risk, applying Proposition 4, we have that $\frac{dq^*}{d\sigma_e} > (<)0$ if and only if $\rho_{\sigma_\pi} > (<)1$. This result is established in Proposition 1 of Broll et al. (2006). Using Proposition 5 of our paper, such a result is equivalent to $(\pi - b)P(\pi) > (<)2$.

Eichner and Wagener (2014) established the relationship between insurance demand and risk aversion elasticity. In this paper, $w = \overline{w} -$

[18] Vergara (2017) extended the results of Gunning (2010) made under the assumption of the CRRA (constant relative risk aversion) utility function and connected his results with the utility premium of Friedman and Savage (1948).

[19] Rothschild and Stiglitz (1971) consider the special case when $y = 0$. Thus, a relative prudence greater than 2 guarantees a positive precautionary effect.

Table 2
Second-order risk: σ_θ versus τ.

Type of risk	$\frac{ds^*}{d\sigma_\theta}$	$\frac{ds^*}{d\tau}$	General prudence index	Elasticity of risk aversion
Labor income risk	+(−)	+(−)	$u'''(c_2) > (<)0$	$\nexists$
Wealth risk	+(−)	+(−)	$c_2P(c_2) > (<)2$	$\rho_{\sigma_{c_2}} > (<)[1 + \frac{\sigma_{c_2}}{\sigma_\theta((1-\delta)+h'(s))} \frac{U_{\mu\sigma}}{U_\sigma} \frac{V_\mu}{U_\mu}]$
Asset risk	+(−)	+(−)	$(c_2 - b)P(c_2) > (<)2$	$\rho_{\sigma_{c_2}} > (<)[1 + \frac{U_{\mu\sigma}}{U_\sigma} \frac{V_\mu}{U_\mu} s]$
Capital income risk	+(−)	+(−)	$(c_2 - b)P(c_2) > (<)2$	$\rho_{\sigma_{c_2}} > (<)[1 + \frac{U_{\mu\sigma}}{U_\sigma} \frac{V_\mu}{U_\mu} \frac{h(s)}{h'(s)}]$
Interest rate risk	+(−)	+(−)	$(c_2 - y)P(c_2) > (<)2$	$\rho_{\sigma_{c_2}} > (<)[1 + \frac{U_{\mu\sigma}}{U_\sigma} \frac{V_\mu}{U_\mu} s]$

$(1-\alpha)z - \alpha p$ represents the final wealth of an individual, who has initial wealth $\overline{w}$ and faces an insurable risky loss of amount z. The control variable α denotes the coinsurance rate, and p is the marginal cost of the insurance purchase. For this case, the mean is $\mu_w = \overline{w} - \mu_z - \alpha(p - \mu_z)$, while the standard deviation is $\sigma_w = (1 - \alpha)\sigma_z$. Applying Proposition 1, we find that the demand for insurance increases (decreases) with a lower expected value of the insurance loss if and only if $\rho_{\mu_w} < (>) - \frac{\mu_w}{\mu_w - b}$. Equivalently, applying Proposition 2, we have $\frac{d\alpha^*}{d\mu_z} < (>)0$ if and only if $(w - b)A(w) > (<)1$, where $b = \overline{w} - p$. Note that if $p = (1 + \lambda)\mu_z$, where λ is a fixed loading factor, then $\mu_w = \overline{w} - (1 + \alpha\lambda)\mu_z$ and $b = \overline{w}$. These results correspond to results 1 and 3 in Eichner and Wagener (2014). In the context of second-order risk, applying Proposition 4, we have that $\frac{d\alpha^*}{d\sigma_z} > (<)0$ if and only if $\rho_{\sigma_w} > (<)1$. This result is equivalent, applying Proposition 5, to $(w - b)P(w) > (<)2$ (Eichner and Wagener, 2004a, 2004b).

Finally, we analyze the banking firm model developed in Broll et al. (2015). In this paper, the bank owners receive the final wealth $W = (r - r_c)A - C(A)$, where A denotes the financial assets of the bank, r represents the financial asset return risk and r_c denotes the bank's weighted average cost of capital. In this model, the mean is $\mu_W = (\mu_r - r_c)A - C(A)$, while the standard deviation is $\sigma_W = \sigma_r A$. It is straightforward to show, applying Proposition 1, that $\frac{dA^*}{d\mu_r} < (>)0$ if and only if $\rho_{\mu_W} < (>) - \frac{\mu_W}{\mu_W - b} = -\frac{1}{R\sigma_W/\mu_W}$, where $R = -\frac{U_\sigma}{U_\mu}$ and $b = A[C'(A) - \frac{C(A)}{A}]$. This result corresponds to Proposition 2 in Broll et al. (2015).[20] Using Proposition 2, this result is equivalent to $(W - b)A(W) > (<)1$. In the context of second-order risk, applying Proposition 4, we have that $\frac{dA^*}{d\sigma_r} > (<)0$ if and only if $\rho_{\sigma_W} > (<)1$. This result is established in Proposition 1 in Broll et al. (2015).[21] Using Proposition 5 of our paper, this result is equivalent to $(W - b)P(W) > (<)2$.

7. Conclusion

The study of the consequences of risk and uncertainty shocks and their effects on the economy has become extremely important in the past year. The economic effects of the COVID-19 pandemic are still unknown because the extent of the damage in the productive sector, the time and shape of the recovery and the strength of the negative expectations of economic agents are still unknown variables. Additionally, political and social unrest has started to appear in places that, up until recently, were considered examples of politically stable countries with decades of good economic performance. In consequence, a deeper look at the effect of uncertainty in the economy is both necessary and important.

We focus this work on how uncertainty affects precautionary saving. We first point out an inconsistency found in the literature, where most of the theoretical work is developed in a dynamical setting under the expected utility paradigm. However, most of the empirical work is based on measures of volatility like variance or standard deviation, which are atheoretical statistical moments that are not related with optimization behavior in decision problems under uncertainty. We then provide a theoretical foundation for using variance as a measure of uncertainty in the analysis of precautionary saving. To this end and following the techniques provided by Sinn (1983) and Meyer (1987), we use the mean-variance approach to analyze the demand for precautionary saving. Unlike the precautionary premium, the variance is compatible with different sources of risk, and we exploit this flexibility in the paper. We use the elasticity of risk aversion to study the effect of first- and second-order risk increases on the demand for precautionary saving. We show that the threshold for the comparison of the consumption mean-related or standard deviation-related elasticity of risk aversion is not equal to unity or a constant, as in other problems in economics under the expected utility paradigm, but that the threshold depends on the level of saving.

Our results in the mean-variance space have counterparts in two general indices derived in the expected utility space. We derive reduced-form indices for the following five sources of risk: labor income risk, wealth risk, asset risk, capital income risk and interest rate risk, and we show the link between these results in both spaces (expected utility and mean-variance), advancing the theory of saving another step forward within the literature.

Our theoretical work not only allows us to intellectually better understand how uncertainty affects saving, but it also provides a more solid ground for public policy aiming to spur investment and consumption to promote economic recovery during the current pandemic shock. Understanding that different sources of uncertainty have different effects on total saving and consumption is key to implement the right policy instrument and so the ideas developed in this paper help to highlight such differences, providing a closer look at the effect of uncertainty on economic agents and their behavior.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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[20] In Broll et al. (2015), the result is $\frac{dA^*}{d\mu_r} < (>)0$ if and only if $\rho_{\mu_W} > (<) \frac{\mu_W}{\mu_W - b} = \frac{1}{R\sigma_W/\mu_W}$ because risk-aversion elasticity is defined as $\rho_\mu = \frac{\partial R}{\partial \mu} \frac{\mu}{R}$.

[21] In Broll et al. (2015), the result is $\frac{dA^*}{d\sigma_r} > (<)0$ if and only if $\rho_{\sigma_W} < (>) - 1$ because risk-aversion elasticity is defined as $\rho_\sigma = \frac{\partial R}{\partial \sigma} \frac{\sigma}{R}$.

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