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*Research Article*

# Dirichlet Type Problem for 2D Quaternionic Time-Harmonic Maxwell System in Fractal Domains

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We investigate an electromagnetic Dirichlet type problem for the 2D quaternionic time-harmonic Maxwell system over a great generality of fractal closed type curves, which bound Jordan domains in $\mathbb{R}^2$. The study deals with a novel approach of $h$-summability condition for the curves, which would be extremely irregular and deserve to be considered fractals. Our technique of proofs is based on the intimate relations between solutions of time-harmonic Maxwell system and those of the Dirac equation through some nonlinear equations, when both cases are reformulated in quaternionic forms.

## 1. Introduction

A theory of hyperholomorphic functions of two real variables is the most natural and close generalization of complex analysis that preserves many of its important features. Some integral representation type formulas in closed Jordan rectifiable curves are proved in [1, 2] and [3, Appendix 4]. Applications in physical problems with elliptic geometries and potential theory can be found in [4–6].

The Maxwell equations govern the behavior of the electromagnetic field. Despite the fact that these equations are more than hundred years old, they still are subject to changes in content, notation, and frameworks.

The quaternionic analysis gives us a tool of wider applicability for the study of electromagnetic boundary value problems. In particular, a quaternionic hyperholomorphic approach to time-harmonic solutions of the Maxwell system is established in [3, 7–9] and the references given there.

These studies confine attention to Lipschitz domains in the worst case scenario. For pure and applied mathematical interest, in [10] some boundary value problems for time-harmonic electromagnetic fields on the more challenging case of domains with fractal boundaries are discussed. Results concerning boundary value problems for the time-harmonic Maxwell system along classical lines can be found in [11]. An overview of different methods that are useful in the analysis of the time-harmonic Maxwell equations was given in [12].

The main goal of this work is the study of an electromagnetic Dirichlet type problem for a domain with fractal boundary in $\mathbb{R}^2$. For a deeper discussion of some electromagnetic problems in two dimensions, we refer the reader to [13–16].

Our main motivation for the introduction of quaternionic analysis in electromagnetics is the difficulty in solving the Maxwell equations in fractal domains involving boundary condition on fractal boundaries, which requires the use of very advanced mathematical techniques.

The outline of the paper is as follows: In Section 2 we provide an outlook to the basics of quaternionic analysis and elements of fractals geometry; a new hyperholomorphic Cauchy type integral for a domain with $h$–summable boundary in $\mathbb{R}^2$ is described in Section 3, where we also state theoretical results on integral representation formulas in domains bounded by such curves. Results on jump boundary value problems across an $h$-summable boundaries of domains in $\mathbb{R}^2$, as well as certain Dirichlet type problems for hyperholomorphic solutions of two dimensional Helmholtz equation are presented and discussed in Section 4. Finally, Section 5 analyzes some Dirichlet type problems involving electromagnetism in the form of 2D quaternionic time-harmonic Maxwell system on a domain with $h$-summable boundary in $\mathbb{R}^2$.

## 2. Preliminaries

The noncommutative and associative algebra with zero divisors of the complex quaternions is denoted by $\mathbb{H}(\mathbb{C})$. For each complex quaternion $a$, one has $a = \sum_{k=0}^{3} a_k e_k$ where $\{a_k\} \subset \mathbb{C}$, $e_0$ is the multiplicative unit and $\{e_k | k = 1,2,3\}$ are standard quaternionic imaginary units. By definition, the complex imaginary unit $i$ satisfies

$$ie_k = e_k i, \quad k = 0,1,2,3. \tag{1}$$

Let $a = a_0 + \overrightarrow{a} = \sum_{k=0}^{3} a_k e_k$, where $a_0 =: Sc(a)$ is called scalar part and $\overrightarrow{a} =: Vec(a)$ is called vector part of the quaternion $a$. The module of $a$ coincides with its Euclidean norm: $|a| = \|a\|_{\mathbb{R}^8}$ and the quaternionic conjugate is defined by $\overline{a} = a_0 - \overrightarrow{a}$. If $Sc(a) = 0$ then $a = \overrightarrow{a}$ is called a purely vector quaternion.

The multiplication of two quaternions $a, b$ can be rewritten in vector terms:

$$ab = a_0 b_0 - \overrightarrow{a} \cdot \overrightarrow{b} + a_0 \overrightarrow{b} + b_0 \overrightarrow{a} + \overrightarrow{a} \times \overrightarrow{b}, \tag{2}$$

where $\overrightarrow{a} \cdot \overrightarrow{b}$ and $\overrightarrow{a} \times \overrightarrow{b}$ are the scalar and the usual cross product in $\mathbb{R}^2$ respectively. We shall frequently write $z := (x, y)$ for a typical point of $\mathbb{R}^2$.

Let a domain $\Omega \subset \mathbb{R}^2$, we will consider $\mathbb{H}(\mathbb{C})$-valued functions:

$$f : \Omega \to \mathbb{H}(\mathbb{C}). \tag{3}$$

Properties of continuity, differentiability and integrability of $f$ have to be understood component-wise. The set of $k$ times continuously differentiable functions is denoted by $C^k(\Omega; \mathbb{H}(\mathbb{C}))$, $k \in \mathbb{N} \cup \{0\}$.

Given $\lambda \in \mathbb{C} \setminus \{0\}$, let $\alpha \in \mathbb{H}(\mathbb{C})$ such that $\alpha^2 = \lambda$. This $\lambda$ generates the (left and right) 2D Helmholtz operator, which acting on $C^2(\Omega; \mathbb{H}(\mathbb{C}))$ are given by ${}_{\lambda}\Delta := \Delta_{\mathbb{R}^2} + {}^{\lambda}M$ and $\Delta_{\lambda} := \Delta_{\mathbb{R}^2} + M^{\lambda}$ respectively. Here and subsequently, $\Delta_{\mathbb{R}^2} := \partial_1^2 + \partial_2^2$, $\partial_k := \partial \backslash \partial x_k$ and for $\lambda \in \mathbb{H}(\mathbb{C})$, $M^{\lambda}[f] := f\lambda$ and ${}^{\lambda}M[f] := \lambda f$.

Additionally, the following partial differential operators will be considered

$$\begin{aligned} {}_{st}\partial &:= e_1 \partial_1 + e_2 \partial_2; \\ {}_{st}\overline{\partial} &:= \overline{e}_1 \partial_1 + \overline{e}_2 \partial_2; \end{aligned} \tag{4}$$

$$\begin{aligned} \partial_{st} &:= \partial_1 \circ M^{e_1} + \partial_2 \circ M^{e_2}; \\ \overline{\partial}_{st} &:= \partial_1 \circ M^{\overline{e}_1} + \partial_2 \circ M^{\overline{e}_2}. \end{aligned} \tag{5}$$

It follows easily that,

$${}_{st}\partial^2 = \partial_{st}^2 = -\Delta_{\mathbb{R}^2}. \tag{6}$$

Set ${}_{\alpha}\partial := \partial_{st} + {}^{\alpha}M$; $\partial_{\alpha} := {}_{st}\partial + M^{\alpha}$. Therefore, the Helmholtz operator can be factorized as follows:

$$\Delta_{\lambda} = -\partial_{\alpha} \circ \partial_{-\alpha} = -\partial_{-\alpha} \circ \partial_{\alpha}. \tag{7}$$

*Definition 1.* A function $f \in C^1(\Omega, \mathbb{H}(\mathbb{C}))$ is called hyperholomorphic if it satisfies $\partial_{\alpha}[f] \equiv 0$ on $\Omega$.

If we write $f = \sum_{k=0}^{3} f_k e_k = f_0 + \overrightarrow{f}$, then we obtain by straightforward calculation

$${}_{st}\partial[f] = -\operatorname{div}\overrightarrow{f} + \operatorname{grad} f_0 + \operatorname{rot}\overrightarrow{f}. \tag{8}$$

From [17], if $\lambda = \alpha^2 \in \mathbb{C}$, a fundamental solution $\theta_{\alpha}$ of $\Delta_{\lambda}$ is given by

$$\theta_{\alpha}(z) := \begin{cases} (-1)^s \dfrac{i}{4} H_0^{(s)}(\alpha|z|), & \text{if} \quad \alpha \neq 0, \\ \dfrac{1}{2\pi} \log|z|, & \text{if} \quad \alpha = 0, \end{cases} \tag{9}$$

where

$$s := \begin{cases} 1, & \text{if} \quad \operatorname{Im}(\alpha) > 0 \text{ or } \alpha > 0, \\ 2, & \text{if} \quad \operatorname{Im}(\alpha) < 0 \text{ or } \alpha < 0, \end{cases} \tag{10}$$

and $H_n^{(s)}$ is the Hankel function of the kind $s$ and of order $n \in \{0, 1, 2\}$ (see [18]).

If $\alpha \neq 0$, $\operatorname{Im}(\alpha) = 0$ the functions $-(i/4)H_0^{(1)}(\alpha|z|)$ and $(i/4)H_0^{(2)}(\alpha|z|)$ are fundamental solutions of the Helmholtz operator $\Delta_{\alpha^2}$.

By (7), the fundamental solution of the operator $\partial_{\alpha}$, i.e., the quaternionic Cauchy kernel, is defined as

$$\mathcal{K}_{st,\alpha}(z) := -\partial_{-\alpha}\theta_{\alpha}(z), \quad z \in \mathbb{R}^2 \setminus \{0\}. \tag{11}$$

Hence

$$\mathcal{K}_{st,\alpha}(z) = \begin{cases} (-1)^s \dfrac{i\alpha}{4} \left( H_1^{(s)}(\alpha|z|) \dfrac{z}{|z|} + H_0^{(s)}(\alpha|z|) \right), & \text{if } \alpha \neq 0, \\ -\dfrac{z}{2\pi|z|^2}, & \text{if } \alpha = 0. \end{cases} \tag{12}$$

*Remark 2.1.* In what follows we suppose that $\alpha = \alpha_0 \in \mathbb{C}$.

Let us now take a quick look at the notion of majorant, with the purpose of considering the generalized Hölder spaces, see [19, 20]. Let $\phi$ be a continuous increasing function on $[0, \infty)$ such that $\phi(0) = 0$ and $\phi(t)/t$ is nonincreasing. One such function $\phi$ said to be a majorant. Note that $\phi(t) = t^{\nu}$, $0 < \nu < 1$, are majorants.

In what follows $c$ will denote a positive constant, not necessarily the same at different occurrences.

Suppose $\mathbf{E} \subset \mathbb{R}^2$ be a bounded set. The generalized Hölder space, denoted by $C^{0,\phi}(\mathbf{E}, \mathbb{H}(\mathbb{C}))$, is defined to be the family of all $\mathbb{H}(\mathbb{C})$-valued functions $f$ on $\mathbf{E}$ such that

$$\left| f(x) - f(y) \right| \leq c\phi(|x - y|), \quad x, y \in \mathbf{E}, \tag{13}$$

where $\phi$ is a given majorant. For $\phi(t) = t^{\nu}$, $0 < \nu < 1$, we write $C^{0,\nu}(\mathbf{E}, \mathbb{H}(\mathbb{C}))$ instead of $C^{0,\phi}(\mathbf{E}, \mathbb{H}(\mathbb{C}))$.

We recall that a nonnegative and almost increasing (or almost decreasing) function $\phi$ means that there exists $c \geq 1$ such that $\phi(x) \leq c\phi(y)$ for all $x \leq y$ $(y \leq x)$, respectively.

Following [21, Definition 1.1], we say that a majorant $\phi$ has order $\nu_{\phi}$ if there exists a $\nu_{\phi}$ $(0 < \nu_{\phi} < 1)$ and a positive real number $t_0$ such that

$$\begin{aligned} \nu_{\phi} &= \sup\left\{ \nu : \frac{\phi(t)}{t^{\nu}} \text{ is almost increasing, } 0 < t < t_0 \right\} \\ &= \inf\left\{ \nu : \frac{\phi(t)}{t^{\nu}} \text{ is almost decreasing, } 0 < t < t_0 \right\}. \end{aligned} \tag{14}$$

This guarantees that $c^{-1}t^{\nu_\phi} \le \phi(t) \le ct^{\nu_\phi}$, $0 < t < t_0$.

If a majorant $\phi$ has order $\nu_\phi$, then we let $\phi = \phi_\nu$ and we will use the symbol $C^{0,\phi_\nu}(\mathbf{E}, \mathbb{H}(\mathbb{C}))$ instead $C^{0,\phi}(\mathbf{E}, \mathbb{H}(\mathbb{C}))$.

The Whitney extension theorem (see [22]) in the quaternionic analysis context is stated as follows.

**Theorem 1.** *Suppose $\mathbf{E} \subset \mathbb{R}^2$ be a compact set and let $f \in C^{0,\varphi_\nu}(\mathbf{E}, \mathbb{H}(\mathbb{C}))$. There exists a compactly supported function $\tilde{f}$ such that*

(i) $\tilde{f}\big|_{\mathbf{E}} = f$;
(ii) $\tilde{f} \in C^\infty(\mathbb{R}^2 \backslash \mathbf{E})$;
(iii) $\left|{}_{st}\partial[\tilde{f}](x,y)\right| \le c(\varphi_\nu(dist(z,\mathbf{E}))/(dist(z,\mathbf{E})))$, $z = (x,y) \in \mathbb{R}^2 \backslash \mathbf{E}$.

*The function $\tilde{f}$ is called a Whitney type extension of $f$. With the notation $dist(A,B)$ we stand for the distance between the subsets $A$ and $B$ of $\mathbb{R}^2$.*

*2.1. Elements of Fractal Geometry.* Let $h : (0,\infty) \to (0,\infty)$ be a gauge function, i.e., a continuous and nondecreasing interval function with $\lim_{t\to 0+} h(t) = 0$.

In [23] a variation of the geometric concept of $d$-summability, which is due to Harrison and Norton in [24] is introduced.

*Definition 2* [23, Definition 1]. Let h be a gauge function. The set $\mathbf{E}$ is called $h$-summable if there exists $\delta > 0$ such that

$$\int_0^\delta N_{\mathbf{E}}(t) \frac{h(t)}{t} dt < +\infty, \tag{15}$$

where $N_{\mathbf{E}}(t)$ stands for the least number of open balls of radius $t$ needed to cover $\mathbf{E}$. When $h(t) = t^d$ with $d \in (1,2)$, we recover the $d$-summability of $\mathbf{E}$.

Definition 2 is unchanged if $N_{\mathbf{E}}(t)$ with $2^{-k} \le t < 2^{-k+1}$ is replaced by the number of $k$-squares intersecting $\mathbf{E}$. By a $k$ -square we mean one of the form

$$\left[l_1 2^{-k}, (l_1+1)2^{-k}\right] \times \left[l_2 2^{-k}, (l_2+1)2^{-k}\right], \tag{16}$$

where $k, l_1, l_2$ are integers.

We follow [22] considering the Whitney decomposition of $\Omega$

$$\Omega = \bigcup_{k=1}^{+\infty} \mathcal{W}^k =: \mathcal{W} = \bigcup_{k=1}^{+\infty} \bigcup_{Q\in\mathcal{W}^k} Q \equiv \bigcup_{Q\in\mathcal{W}} Q. \tag{17}$$

The squares $Q$ in $\mathcal{W}$ have disjoint interiors and satisfy

$$dist(z,\Gamma) \ge \frac{1}{\sqrt{2}}|Q| = 2^{-k+1}, \quad z \in Q, Q \in \mathcal{W}^k. \tag{18}$$

Here and subsequently, $|\mathbf{E}|$ stands for the diameter of a bounded set $\mathbf{E} \subset \mathbb{R}^2$.

## 3. Hyperholomorphic Cauchy Type Integral for Fractal Curves

The Cauchy type integral associated to quaternionic analysis has been involved recently with fractional metric dimensions and fractals, see [9, 10, 25]. In this section, we define and characterize the hyperholomorphic Cauchy type integral on fractal type curves. Before giving the definition, we will state some preliminary results.

Let $L_p(\Omega, \mathbb{H}(\mathbb{C}))$, $p > 1$, the set of $p$-integrable functions, the Teodorescu transform $T_{\alpha_0}[f]$ for $f \in L_p(\Omega, \mathbb{H}(\mathbb{C}))$, is given by

$$T_{\alpha_0}[f](x,y) := \int_\Omega \mathcal{K}_{st,\alpha_0}(x-u, y-v) f(u,v) du \wedge dv, \quad (x,y) \in \mathbb{R}^2. \tag{19}$$

Looking at the kernel function $\mathcal{K}_{st,\alpha_0}(z)$ we can decompose it in the following way

$$\mathcal{K}_{st,\alpha_0}(z) = \mathcal{K}_{st,0}(z) + \mathcal{K}^1_{st,\alpha_0}(z) + \mathcal{K}^2_{st,\alpha_0}(z), \tag{20}$$

where $\mathcal{K}^1_{st,\alpha_0}(z) := (\alpha_0/(2\pi)) \ln|z|$ and the continuous function $\mathcal{K}^2_{st,0}(z) = 0$.

*Remark 3.1.* It is to be expected that $T_{\alpha_0}$ shares many of the properties of $T_0$. For example, $T_{\alpha_0}[f] \in C^{0,(p-2)/p}(\mathbb{R}^2, \mathbb{H}(\mathbb{C}))$, if $f \in L_p(\Omega, \mathbb{H}(\mathbb{C}))$ for $p > 2$, by analogy with [7, Subsection 8.1].

In what follows, given $\varphi_\nu$ a majorant and $d \in (1,2)$, we will take $\Omega \subset \mathbb{R}^2$ to be a Jordan domain with $h$-summable boundary $\Gamma$, for $h(t) = t^{d-1}\varphi_\nu(t)$ and $t \in (0, |\Gamma|]$.

*Definition 3.* We define the hyperholomorphic Cauchy type integral of $f \in C^{0,\varphi_\nu}(\Gamma, \mathbb{H}(\mathbb{C}))$ by the formula

$$K^*_{\alpha_0}[f](x,y) = \chi_\Omega \tilde{f}(x,y) - T_{\alpha_0} \circ \partial_{\alpha_0}[\tilde{f}](x,y), \quad (x,y) \in \mathbb{R}^2 \backslash \Gamma, \tag{21}$$

where $\chi_\Omega$ is the indicator function of $\Omega$.

The following proposition makes Definition 3 legitimate.

**Proposition 1.** *The integral (21) is independent of the choice of $\tilde{f}$.*

*Proof.* By definition, $\partial_{\alpha_0}[\tilde{f}] = {}_{st}\partial[\tilde{f}] + M^{\alpha_0}[\tilde{f}]$. As $\tilde{f} \in C^{0,\varphi_\nu}(\Omega \cup \Gamma, \mathbb{H}(\mathbb{C}))$ we have $M^{\alpha_0}[\tilde{f}] \in L_p(\Omega, \mathbb{H}(\mathbb{C}))$ for any $p > 0$. The proof is completed by showing that ${}_{st}\partial[\tilde{f}] \in L_1(\Omega, \mathbb{H}(\mathbb{C}))$.

We have

$$\int_\Omega \left|{}_{st}\partial[\tilde{f}](x,y)\right| dx \wedge dy = \sum_{Q\in\mathcal{W}} \int_Q \left|{}_{st}\partial[\tilde{f}](x,y)\right| v dx \wedge dy \le c \cdot \sum_{Q\in\mathcal{W}} \int_Q \frac{\varphi_\nu(dist(z,\Gamma))}{dist(z,\Gamma)} dx \wedge dy, \tag{22}$$

which is a consequence of Theorem 1 (iii).

According to (18) and taking account that $\varphi_\nu(t)/t$ does not increase, we have

$$\frac{\varphi_\nu(dist(z,\Gamma))}{dist(z,\Gamma)} \le \frac{\varphi_\nu(dist(Q,\Gamma))}{dist(Q,\Gamma)} \le \frac{\varphi_\nu(|Q|)}{|Q|}, \quad z \in Q. \tag{23}$$

The inequality (23) and the fact that $d-1<1$, gives

$$\begin{aligned}\sum_{Q\in\mathcal{W}}\int_Q \frac{\varphi_\nu(dist(z,\Gamma))}{dist(z,\Gamma)}dx\wedge dy &\le \sum_{Q\in\mathcal{W}}\int_Q \frac{\varphi_\nu(|Q|)}{|Q|}dx\wedge dy\\ &= \sum_{Q\in\mathcal{W}}|Q|\varphi_\nu(|Q|)\\ &\le \sum_{Q\in\mathcal{W}}|Q|^{d-1}\varphi_\nu(|Q|).\end{aligned} \tag{24}$$

Consequently

$$\int_\Omega \left|{}_{st}\partial\left[\widetilde{f}\right](x,y)\right|dx\wedge dy \le c\sum_{Q\in\mathcal{W}}|Q|^{d-1}\varphi_\nu(|Q|), \tag{25}$$

where the last sum is finite due to [23, Lemma 1].

Now suppose that $\widetilde{f}$ and $\widetilde{g}$ are two different Whitney type extensions of $f$. Then $\widetilde{l}:=\widetilde{f}-\widetilde{g}$, is a Whitney type extension of the null function and hence $\widetilde{l}|_\Gamma=0$. It remains to prove that

$$\chi_\Omega\widetilde{l}(x,y)-T_{\alpha_0}\circ\partial_{\alpha_0}\left[\widetilde{l}\right](x,y)=0,\quad (x,y)\in\mathbb{R}^2\backslash\Gamma. \tag{26}$$

To this end, let us consider the following connected domains

$$\Omega_k:=\left\{z\in Q\mid Q\in\mathcal{W}^j,\ \text{for some } j\le k\right\}. \tag{27}$$

The boundary of $\Omega_k$, denoted by $\Gamma_k$, consists of sides of some squares $Q\in\mathcal{W}^k$.

Thus, we have

$$\begin{aligned}&\int_\Omega \mathcal{K}_{st,\alpha_0}(x-u,y-v)\partial_{\alpha_0}\left[\widetilde{l}\right](u,v)du\wedge dv\\ &\quad=\lim_{k\to\infty}\int_{\Omega_k}\mathcal{K}_{st,\alpha_0}(x-u,y-v)\partial_{\alpha_0}\left[\widetilde{l}\right](u,v)\,du\wedge dv.\end{aligned} \tag{28}$$

Now, take $z\in\Omega$ and choose $k_0$ sufficiently large such that $z\in\Omega_{k_0}$ and for $k>k_0$, $dist(z,\Gamma_k)\ge|Q_0|/2\sqrt{2}$, where $Q_0$ is a square of $\mathcal{W}^{k_0}$. The quaternionic Borel–Pompeiu formula, see [4, Theorem 4.1, Theorem 4.4], applied to $\Omega_k$, yields

$$\begin{aligned}&\widetilde{l}(x,y)+\int_{\Omega_k}\mathcal{K}_{st,\alpha_0}(x-u,y-v)\partial_{\alpha_0}\left[\widetilde{l}\right](u,v)du\wedge dv\\ &\quad=\int_{\Gamma_k}\mathcal{K}_{st,\alpha_0}(x-u,y-v)n_k(u,v)\widetilde{l}(u,v)\,d\Gamma_{(u,v)},\end{aligned} \tag{29}$$

where $n_k(u,v)$ is the unit normal vector on $\Gamma_k$ and $d\Gamma_{(u,v)}$ denotes the surface measure. Next, let $w:=(u+iv)\in\Gamma_k$, $Q\in\mathcal{W}^k$ a square containing $w$, and $z\in\Gamma$ such that $|w-z|=dist(w,\Gamma)$. Since $\widetilde{l}|_\Gamma=0$, it follows that

$$\left|\widetilde{l}(w)\right|=\left|\widetilde{l}(w)-\widetilde{l}(z)\right|\le c\varphi_\nu(|w-z|)\le c\varphi_\nu(|Q|). \tag{30}$$

If $\Sigma$ is a side of $\Gamma_k$ and $Q\in\mathcal{W}^k$ is the $k$-square containing $\Sigma$, we have for $k>k_0$

$$\begin{aligned}&\left|\int_\Sigma \mathcal{K}_{st,\alpha_0}(x-u,y-v)n_k(u,v)\widetilde{l}(u,v)\,d\Gamma_{(u,v)}\right|\\ &\quad\le\frac{c}{|Q_0|}\int_\Sigma\left|\widetilde{l}(u,v)\right|d\Gamma_{(u,v)}\le\frac{c}{|Q_0|}|Q|\varphi_\nu(|Q|).\end{aligned} \tag{31}$$

Because each side of $\Gamma_k$ belongs to some $Q\in\mathcal{W}^k$, we have for $k>k_0$,

$$\begin{aligned}&\left|\int_{\Gamma_k}\mathcal{K}_{st,\alpha_0}(x-u,y-v)n_k(u,v)\widetilde{l}(u,v)\,d\Gamma_{(u,v)}\right|\\ &\quad\le\frac{c}{|Q_0|}\sum_{Q\in W^k}|Q|\varphi_\nu(|Q|)\le\frac{c}{|Q_0|}\sum_{Q\in W^k}|Q|^{d-1}\varphi_\nu(|Q|).\end{aligned} \tag{32}$$

The conclusion of [23, Lemma 1], implies that

$$\lim_{k\to\infty}\int_{\Gamma_k}\mathcal{K}_{st,\alpha_0}(x-u,y-v)n_k(u,v)\widetilde{l}(u,v)d\Gamma_{(u,v)}=0. \tag{33}$$

Combining (28) with (29) yields (26) for $w\in\Omega$. The case $w\in\mathbb{R}^2\backslash\{\Omega\cup\Gamma\}$ can be handled in the same way; the difference is in the fact that now $dist(w,\Gamma_k)\ge dist(w,\Gamma)$. ☐

In the rest of this paper we assume $\Omega_+:=\Omega$ and $\Omega_-:=\mathbb{R}^2\backslash\{\Omega_+\cup\Gamma\}$.

*3.1. Integral Representation Formulae.* The following formulas represent extensions to those given in [25] for the case of a Jordan domain with a $d$-summable boundary.

**Theorem 2** (Borel–Pompeiu formula). *Suppose $f\in C^1(\Omega_+\cup\Gamma,\mathbb{H}(\mathbb{C}))$, then*

(i) $K^*_{\alpha_0}[f](x,y)+T_{\alpha_0}\circ\partial_{\alpha_0}[f](x,y)=\begin{cases}f(x,y), & (x,y)\in\Omega_+\\ 0, & (x,y)\in\Omega_-\end{cases}$

(ii) $\partial_{\alpha_0}\circ T_{\alpha_0}[f](x,y)=f(x,y),\quad (x,y)\in\Omega_+$

*hold.*

Let us mention two important consequences of Theorem 2.

**Theorem 3** (Koppelman formula). *Let f satisfy the hypotheses of above theorem, then the following equality holds*

$$\begin{aligned}&K^*_{\alpha_0}[f](x,y)+T_{\alpha_0}\circ\partial_{\alpha_0}[f](x,y)+\partial_{\alpha_0}\circ T_{\alpha_0}[f](x,y)\\ &\quad=\begin{cases}2f(x,y), & (x,y)\in\Omega_+\\ 0, & (x,y)\in\Omega_-.\end{cases}\end{aligned} \tag{34}$$

**Theorem 4** (Cauchy formula). *Let $f\in C^1(\Omega_+\cup\Gamma,\mathbb{H}(\mathbb{C}))$ such that be hyperholomorphic in $\Omega_+$, then*

$$K^*_{\alpha_0}[f](x,y)=\begin{cases}f(x,y), & (x,y)\in\Omega_+\\ 0, & (x,y)\in\Omega_-.\end{cases} \tag{35}$$

Let us now establish and prove the following auxiliary lemma.

**Lemma 1.** *Let $f\in C^{0,\varphi_\nu}(\Gamma,\mathbb{H}(\mathbb{C}))$, then $\partial_{\alpha_0}\left[\widetilde{f}\right]\in L_p(\Omega_+,\mathbb{H}(\mathbb{C}))$ for all*

$$p<\frac{3-\nu_\varphi-d}{1-\nu_\varphi}. \tag{36}$$

*Proof.* The proof will be divided into two steps. First $M^{\alpha_0}\left[\widetilde{f}\right]\in L_p(\Omega_+,\mathbb{H}(C))$ for any $p>0$, which follows from the fact that $\widetilde{f}\in C^{0,\varphi_\nu}(\Omega_+\cup\Gamma,\mathbb{H}(C))$ with $\Omega_+$ bounded. The next step is to prove that ${}_{st}\partial\left[\widetilde{f}\right]\in L_p(\Omega_+,\mathbb{H}(\mathbb{C}))$ for $p<\left(3-\nu_\varphi-d\right)/\left(1-\nu_\varphi\right)$. Indeed, application of Theorem 1 (iii) enables us to write

$$\int_\Omega \left|{}_{st}\partial\left[\tilde{f}\right](x,y)\right|^p dx\wedge dy = \sum_{Q\in\mathcal{W}}\int_Q \left|{}_{st}\partial\left[\tilde{f}\right](x,y)\right|^p dx\wedge dy \leq c\sum_{Q\in\mathcal{W}}\int_Q\left[\frac{\varphi_\nu(dist(z,\Gamma))}{dist(z,\Gamma)}\right]^p dx\wedge dy. \tag{37}$$

According to (18) and taking account that $\varphi_\nu(t)/t$ does not increase, we have

$$\frac{\varphi_\nu(dist(z,\Gamma))}{dist(z,\Gamma)} \leq \frac{\varphi_\nu(dist(Q,\Gamma))}{dist(Q,\Gamma)} \leq \frac{\varphi_\nu(|Q|)}{|Q|}, \quad z\in Q. \tag{38}$$

Consequently

$$\sum_{Q\in\mathcal{W}}\int_Q\left[\frac{\varphi_\nu(dist(z,\Gamma))}{dist(z,\Gamma)}\right]^p dx\wedge dy \leq \sum_{Q\in\mathcal{W}}\int_Q\left[\frac{\varphi_\nu(|Q|)}{|Q|}\right]^p dx\wedge dy = \sum_{Q\in\mathcal{W}}\varphi_\nu^p(|Q|)|Q|^{2-p} \leq c\sum_{Q\in\mathcal{W}}|Q|^{p\nu_\varphi}|Q|^{2-p}, \tag{39}$$

which is due to the fact that $\varphi_\nu$ is a majorant of order $\nu_\varphi$.

The inequality $p<\left(3-\nu_\varphi-d\right)/\left(1-\nu_\varphi\right)$ implies that $d<\nu_\varphi(p-1)+3-p$, which provides

$$\sum_{Q\in\mathcal{W}}|Q|^{\nu_\varphi(p-1)+3-p} \leq \sum_{Q\in\mathcal{W}}|Q|^d. \tag{40}$$

Therefore

$$\sum_{Q\in\mathcal{W}}|Q|^{p\nu_\varphi+2-p} \leq \sum_{Q\in\mathcal{W}}|Q|^{d-1}|Q|^{\nu_\varphi} \leq c\sum_{Q\in\mathcal{W}}|Q|^{d-1}\varphi_\nu(|Q|). \tag{41}$$

Combining the inequalities (37), (39), and (41) we can conclude that

$$\int_\Omega\left|{}_{st}\partial\left[\tilde{f}\right](x,y)\right|^p dx\wedge dy \leq c\sum_{Q\in\mathcal{W}}|Q|^{d-1}\varphi_\nu(|Q|) < +\infty. \tag{42}$$

It remains to use that $\partial_{\alpha_0}\left[\tilde{f}\right] = {}_{st}\partial\left[\tilde{f}\right] + M^{\alpha_0}\left[\tilde{f}\right]$. □

*Remark 3.2.* Obviously, $K^*_{\alpha_0}[f]$ is a hyperholomorphic function in $\mathbb{R}^2\backslash\Gamma$, which is clear from Theorem 2 (ii). The question of whether $K^*_{\alpha_0}[f]$ admits continuous extensions (to be denoted by $\left[K^*_{\alpha_0}\right]^{\pm}[f]$) to $\Omega_\pm\cup\Gamma$ will be answered affirmatively in the next theorem. We see at once that $\left[K^*_{\alpha_0}\right]^-[f](\infty)=0$.

**Theorem 5.** *Let $d-1<\nu_\varphi$ and consider $f\in C^{0,\varphi_\nu}(\Gamma,\mathbb{H}(\mathbb{C}))$, then $K^*_{\alpha_0}[f]$ admits continuous extensions to $\Omega_\pm\cup\Gamma$ such that*

$$\left[K^*_{\alpha_0}\right]^+[f](x,y) = f(x,y) - T_{\alpha_0}\circ\partial_{\alpha_0}\left[\tilde{f}\right](x,y), \quad (x,y)\in\Gamma, \tag{43}$$

$$\left[K^*_{\alpha_0}\right]^-[f](x,y) = -T_{\alpha_0}\circ\partial_{\alpha_0}\left[\tilde{f}\right](x,y), \quad (x,y)\in\Gamma \tag{44}$$

*belong to $C^{0,\psi_\beta}(\Gamma,\mathbb{H}(\mathbb{C}))$, whenever*

$$\beta_\psi < \frac{1+\nu_\varphi-d}{3-\nu_\varphi-d}. \tag{45}$$

*Proof.* We have $2<\left(3-\nu_\varphi-d\right)/\left(1-\nu_\varphi\right)$, because $d-1<\nu_\varphi$. Thus we are at liberty to choose $p$ such that

$$2<p<\frac{3-\nu_\varphi-d}{1-\nu_\varphi}. \tag{46}$$

For any such $p$, we conclude that $\partial_{\alpha_0}\left[\tilde{f}\right]\in L_p(\Omega_+,\mathbb{H}(C))$ (see Lemma 1). From Remark 3.1 it follows that the integral term in (21), represents a continuous function in $\mathbb{R}^2$. By the above, $K^*_{\alpha_0}[f]$ admits a continuous extension to $\Omega_+\cup\Gamma$. The inequality (45) implies that $\nu_\varphi$ and $(p-2)/p$ are both greater than $\beta_\psi$ and consequently $\left[K^*_{\alpha_0}\right]^+[f]$ belongs to $C^{0,\psi_\beta}(\Gamma,\mathbb{H}(\mathbb{C}))$. The rest of the proof runs as before. □

The following direct corollary is a refinement of [25, Theorem 6] (also see [26]). We check at once that requirement on $d$ and $\nu$ has been weakened.

**Corollary 1.** *Let $d-1<\nu$ and consider $f\in C^{0,\nu}(\Gamma,\mathbb{H}(\mathbb{C}))$, then $K^*_{\alpha_0}[f]$ admits continuous extensions to $\Omega_\pm\cup\Gamma$ such that its boundary values*

$$\left[K^*_{\alpha_0}\right]^+[f](x,y) = f(x,y) - T_{\alpha_0}\circ\partial_{\alpha_0}\left[\tilde{f}\right](x,y), \quad (x,y)\in\Gamma, \tag{47}$$

$$\left[K^*_{\alpha_0}\right]^-[f](x,y) = -T_{\alpha_0}\circ\partial_{\alpha_0}\left[\tilde{f}\right](x,y), \quad (x,y)\in\Gamma \tag{48}$$

*belong to $C^{0,\beta}(\Gamma,\mathbb{H}(\mathbb{C}))$, whenever $\beta<(1+\nu-d)/(3-\nu-d)$.*

## 4. Boundary Value Problems

We deal with three boundary value problems for hyperholomorphic solutions of the two dimensional Helmholtz equation in a fractal domain of $\mathbb{R}^2$.

**Theorem 6.** *Let $f\in C^{0,\varphi_\nu}(\Gamma,\mathbb{H}(\mathbb{C}))$ with $d-1<\nu_\varphi$, then the jump problem*

$$f = f^+ - f^-, \tag{49}$$

*has a solution explicitly given by*

$$f^\pm = \left[K^*_{\alpha_0}\right]^\pm[f]. \tag{50}$$

*where the hyperholomorphic components $f^\pm\in C^{0,\psi_\beta}(\Omega_\pm,\mathbb{H}(\mathbb{C}))$ whenever $\psi_\beta$ is a majorant with order $\beta_\psi<\left(1+\nu_\varphi-d\right)/\left(3-\nu_\varphi-d\right)$, and moreover $f^-(\infty)=0$.*

*Proof.* It is sufficient to use Theorem 5. □

**Theorem 7.** *Suppose that $G\in C^{0,\varphi_\nu}(\Gamma,\mathbb{H}(\mathbb{C}))$with $d-1<\nu_\varphi$. If there exists $f\in C^{0,\varphi_\nu}(\Omega_+\cup\Gamma,\mathbb{H}(\mathbb{C}))$such that*

$$\begin{aligned}\partial_{\alpha_0}[f] &= 0, \quad \text{in } \Omega_+\\ f &= G, \quad \text{on } \Gamma\end{aligned} \tag{51}$$

*then*

$$\left[K^*_{\alpha_0}\right]^-[G](x,y) = 0, \quad (x,y) \in \Gamma. \tag{52}$$

*On the contrary, if (52) holds, there exists a solution $f \in C^{0,\psi_\beta}(\Omega_+ \cup \Gamma, \mathbb{H}(\mathbb{C}))$ of (51), whenever $\psi_\beta$ is a majorant with order $\beta_\psi < (1+\nu_\varphi - d)/(3-\nu_\varphi - d)$.*

*Proof.* Assume (51) holds, which signifies that $f$ is a Whitney type extension of $G$. Application of Theorem 4 gives $f = K^*_{\alpha_0}[f]$, but $K^*_{\alpha_0}[f] = K^*_{\alpha_0}[G]$ as $f = G$ in $\Gamma$.

Now (52) follows after passage to the limit from inside $\Omega_+$. On the other hand, if (52) holds, our claim follows directly by taking $f = K^*_{\alpha_0}[G]$. Analysis similar to that in the proof of Theorem 5 shows that $f \in C^{0,\psi_\beta}(\Omega_+ \cup \Gamma, \mathbb{H}(\mathbb{C}))$. ☐

**Theorem 8.** *Let $G \in C^{0,\varphi_\nu}(\Gamma, \mathbb{H}(\mathbb{C}))$ with $d-1 < \nu_\varphi$ and $F \in L_p(\Omega_+, \mathbb{H}(\mathbb{C}))(p>2)$. If there exists $f \in C^{0,\varphi_\nu}(\Omega_+ \cup \Gamma, \mathbb{H}(\mathbb{C}))$ a solution of*

$$\begin{aligned} \partial_{\alpha_0}[f] &= F, \quad in\, \Omega_+ \\ f &= G, \quad on\, \Gamma \end{aligned} \tag{53}$$

*then*

$$\left[K^*_{\alpha_0}\right]^-[G](x,y) = -T_{\alpha_0}[F](x,y), \ (x,y) \in \Gamma. \tag{54}$$

*Conversely, if (54) holds, there exists a solution $f \in C^{0,\psi_\beta}(\Omega_+ \cup \Gamma, \mathbb{H}(\mathbb{C}))$ of (53), whenever $\psi_\beta$ is a majorant with order*

$$\beta_\psi < \min\left\{\frac{1+\nu_\varphi - d}{3-\nu_\varphi - d}, \frac{p-2}{p}\right\}. \tag{55}$$

*Proof.* Take $g = f - T_{\alpha_0}[F]$ and let $f$ be a solution of (53). By Theorem 2 (ii), we have that $g$ is a solution of (51) with $G$ replaced by $G - \left[T_{\alpha_0}[F]\right]|_\Gamma \in C^{0,\psi_\beta}(\Gamma, \mathbb{H}(\mathbb{C}))$. The equality.

$$\left[K^*_{\alpha_0}\right]^-\left[G - T_{\alpha_0}[F]\right](x,y) = 0, \ (x,y) \in \Gamma, \tag{56}$$

which is clear from Theorem 7 applied to this case, implies (54). Taking $f = K^*_{\alpha_0}[G] + T_{\alpha_0}[F]$, the second assertion follows directly.

Under the assumptions of Theorem 8 we have that the function $f = K^*_{\alpha_0}[G] + T_{\alpha_0}[F]$ does not depend on the $\widetilde{G}$. Consequently, the following equality holds

$$\begin{aligned} f(x,y) = \widetilde{G}(x,y) &- T_{\alpha_0} \circ \partial_{\alpha_0}\left[\widetilde{G}\right](x,y) \\ &+ T_{\alpha_0}[F](x,y), \quad (x,y) \in \Omega_+. \end{aligned} \tag{57}$$

*Remark 4.1.* For a vector-valued function $f$ in Theorem 8, (57) clearly forces $G$ and $F$ to satisfy

$$Sc\left(T_{\alpha_0} \circ \partial_{\alpha_0}\left[\widetilde{G}\right](x,y)\right) = Sc\left(T_{\alpha_0}[F](x,y)\right). \tag{58}$$

## 5. Main Results

Between 1861 and 1862, J. C. Maxwell published the fundamental papers *"A treatise on electricity and magnetism"* and *"A Dynamical Theory of the Electromagnetic Field"*, where the behavior of electromagnetic fields was described and completely formulated the system of partial differential equations, named after him, which form the foundation of classical electromagnetism, radio-electronics, wave propagation theory, and many other branches of physics and engineering. Since this time, the fact that solutions of the Maxwell system (for time-harmonic fields) can be related to solutions of the Dirac equation, through some nonlinear equations. This has fascinated several generations of physicists and mathematicians in various branches of science, because of their general, even philosophical significance, see for instance [3, 5, 8–10]. To the best of the author's knowledge, this deep relation was properly formulated and documented originally in [27].

We shall focus attention on time-harmonic electromagnetic fields, where all fields vary sinusoidally in time with a single frequency of oscillations $\omega$, i.e., with the dependence on the time as $e^{-i\omega t}$.

$$\begin{aligned} \operatorname{rot} \overrightarrow{H} &= -i\omega\varepsilon\overrightarrow{E} + \overrightarrow{j}, \\ \operatorname{rot} \overrightarrow{E} &= i\omega\mu\overrightarrow{H}, \\ \operatorname{div} \overrightarrow{E} &= \frac{\rho}{\varepsilon}, \\ \operatorname{div} \overrightarrow{H} &= 0. \end{aligned} \tag{59}$$

Here and subsequently, $\overrightarrow{E}$ and $\overrightarrow{H}$ denote the complex amplitudes of the electric respectively magnetic field and $\varepsilon$ and $\mu$ are, respectively, the complex-valued absolute permittivity and permeability of the medium. The current density and the charge density are related by the equality $\operatorname{div} \overrightarrow{j} = i\omega\rho$.

In recent decades, interest in the time-harmonic Maxwell system has never dropped. Works noted in [8, 28–35] are some examples of theses achievements in literature.

The current research is oriented towards a quaternionic reformulation of the Maxwell system (59), which is adapted from [8], and given a more simply algebraical structure in the form

$$\begin{aligned} \partial_{-\alpha_0}\left[\overrightarrow{\vartheta}\right] &= \operatorname{div} \overrightarrow{j} + \alpha_0 \overrightarrow{j} \\ \partial_{\alpha_0}\left[\overrightarrow{\eta}\right] &= -\operatorname{div} \overrightarrow{j} + \alpha_0 \overrightarrow{j}, \end{aligned} \tag{60}$$

where $\overrightarrow{\vartheta} = -i\omega\varepsilon\overrightarrow{E} + \alpha_0\overrightarrow{H}$, $\overrightarrow{\eta} = i\omega\varepsilon\overrightarrow{E} + \alpha_0\overrightarrow{H}$ are purely vectorial $\mathbb{H}(\mathbb{C})$-valued functions and the wave number $\alpha_0 = \omega\sqrt{\varepsilon\mu}$ is chosen such that $\operatorname{Im}\alpha_0 \geq 0$.

This equivalence is the key to obtaining in this section our main results concerning the solvability of an inhomogeneous Dirichlet type problem for 2D quaternionic time-harmonic Maxwell system.

**Theorem 9.** *Let $\rho$ and $\overrightarrow{j}$ belong to $L_p(\Omega_+, \mathbb{H}(\mathbb{C}))$, with $p > 2$. Let $\overrightarrow{e}$ and $\overrightarrow{h}$ be complex vector-valued functions in $C^{0,\varphi_\nu}(\Gamma, \mathbb{H}(\mathbb{C}))$. If there exists $\overrightarrow{E}$ and $\overrightarrow{H}$, both in $C^{0,\varphi_\nu}(\Omega_+ \cup \Gamma, \mathbb{H}(\mathbb{C}))$, satisfying in $\Omega_+$ the system (59) such that on $\Gamma$.*

$$\begin{aligned}\left.\vec{E}\right|_{\Gamma} &= \vec{e}, \\ \left.\vec{H}\right|_{\Gamma} &= \vec{h},\end{aligned} \tag{61}$$

*then we have*

$$T_{-\alpha_0} \circ \partial_{-\alpha_0}\left[-i\omega\varepsilon\tilde{\vec{e}} + \alpha_0\tilde{\vec{h}}\right]\Big|_{\Gamma} = T_{-\alpha_0}\left[\operatorname{div}\vec{j} + \alpha_0\vec{j}\right]\Big|_{\Gamma}, \tag{62}$$

$$Sc\left(T_{-\alpha_0} \circ \partial_{-\alpha_0}\left[-i\omega\varepsilon\tilde{\vec{e}} + \alpha_0\tilde{\vec{h}}\right]\right) = Sc\left(T_{-\alpha_0}\left[\operatorname{div}\vec{j} + \alpha_0\vec{j}\right]\right) \quad in\ \Omega_+ \tag{63}$$

*and*

$$T_{\alpha_0} \circ \partial_{\alpha_0}\left[i\omega\varepsilon\tilde{\vec{e}} + \alpha_0\tilde{\vec{h}}\right]\Big|_{\Gamma} = T_{\alpha_0}\left[-\operatorname{div}\vec{j} + \alpha_0\vec{j}\right]\Big|_{\Gamma}, \tag{64}$$

$$Sc\left(T_{\alpha_0} \circ \partial_{\alpha_0}\left[i\omega\varepsilon\tilde{\vec{e}} + \alpha_0\tilde{\vec{h}}\right]\right) = Sc\left(T_{\alpha_0}\left[-\operatorname{div}\vec{j} + \alpha_0\vec{j}\right]\right) \quad in\ \Omega_+. \tag{65}$$

*Conversely, if (62)–(65) are valid, then*

$$\begin{aligned}\vec{E} = \tilde{\vec{e}} - \frac{1}{2i\omega\varepsilon}\Big\{& i\omega\varepsilon\left(T_{\alpha_0} \circ \partial_{\alpha_0} + T_{-\alpha_0} \circ \partial_{-\alpha_0}\right) \\ &\cdot \left[\tilde{\vec{e}}\right] + \alpha_0\left(T_{-\alpha_0} \circ \partial_{-\alpha_0} - T_{\alpha_0} \circ \partial_{\alpha_0}\right) \\ &\cdot \left[\tilde{\vec{h}}\right] - \left(T_{\alpha_0} + T_{-\alpha_0}\right)\left[\operatorname{div}\vec{j}\right] + \alpha_0\left(T_{\alpha_0} - T_{-\alpha_0}\right)\left[\vec{j}\right]\Big\}\end{aligned} \tag{66}$$

*and*

$$\begin{aligned}\vec{H} = \tilde{\vec{h}} - \frac{1}{2\alpha_0}\Big\{& i\omega\varepsilon\left(T_{\alpha_0} \circ \partial_{\alpha_0} - T_{-\alpha_0} \circ \partial_{-\alpha_0}\right) \\ &\cdot \left[\tilde{\vec{e}}\right] - \alpha_0\left(T_{-\alpha_0} \circ \partial_{-\alpha_0} + T_{\alpha_0} \circ \partial_{\alpha_0}\right)\left[\tilde{\vec{h}}\right] \\ &- \left(T_{\alpha_0} - T_{-\alpha_0}\right)\left[\operatorname{div}\vec{j}\right] + \alpha_0\left(T_{\alpha_0} + T_{-\alpha_0}\right)\left[\vec{j}\right]\Big\},\end{aligned} \tag{67}$$

*are solutions of the system (59) and the boundary conditions (61) are satisfied. Moreover,* $\vec{E}$ *and* $\vec{H}$ *belong to* $C^{0,\psi_\beta}(\Omega_+ \cup \Gamma, \mathbb{H}(\mathbb{C}))$ *if* $\psi_\beta$ *is a majorant with order*

$$\beta_\psi < \min\left\{\frac{1+\nu_\varphi - d}{3-\nu_\varphi - d}, \frac{p-2}{p}\right\}. \tag{68}$$

*Proof.* As a first step, we put $\vec{\vartheta} = -i\omega\varepsilon\vec{E} + \alpha_0\vec{H}$ and $\vec{\eta} = i\omega\varepsilon\vec{E} + \alpha_0\vec{H}$. Moreover, the conditions (61) now read.

$$\begin{aligned}\left.\vec{\vartheta}\right|_{\Gamma} &= -i\omega\varepsilon\vec{e} + \alpha_0\vec{h}, \\ \left.\vec{\eta}\right|_{\Gamma} &= i\omega\varepsilon\vec{e} + \alpha_0\vec{h}.\end{aligned} \tag{69}$$

Next, Theorem 8 applied to $\vec{\vartheta}$ and $\vec{\eta}$, together with the equality

$$\left[K^*_{\alpha_0}\right]^-[f](x,y) = -T_{\alpha_0} \circ \partial_{\alpha_0}\left[\tilde{f}\right](x,y), \tag{70}$$

implies the conditions (62) and (64). Additionally, the vectorial nature of the complex amplitudes and Remark 4.1 yield (63) and (65). If we use the identities

$$\begin{aligned}\vec{E} &= \frac{1}{2i\omega\varepsilon}\left(\vec{\eta} - \vec{\vartheta}\right), \\ \vec{H} &= \frac{1}{2\alpha_0}\left(\vec{\eta} + \vec{\vartheta}\right),\end{aligned} \tag{71}$$

a trivial verification completes the proof.

## 6. Conclusions

This paper established a new hyperholomorphic Cauchy type integral for a domain with fractal boundary in $\mathbb{R}^2$, which plays a remarkable role in the theoretical results on integral representation formulas. Three boundary value problems for hyperholomorphic solutions of a two dimensional Helmholtz equation in a fractal domain of $\mathbb{R}^2$ are studied. They have proven successful in the solution of an inhomogeneous Dirichlet type problem for a 2D quaternionic time-harmonic Maxwell system in a domain with fractal boundary in $\mathbb{R}^2$.

## Data Availability

No data were used to support this study.

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

## Authors' Contributions

All authors contributed equally to this work. All authors read and approved the final manuscript.

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